# Word of Mouth, Noise-driven Volatility, and Public Disclosure*

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**Abstract**

This paper examines firms’ responses to technological innovations that improve investors’ private information. We show that more precise private information imposes an endogenous cost of amplifying supply shocks and increasing price volatility. We study how the firm reacts to such changes and derive a necessary and sufficient condition under which the firm improves its disclosure quality when its investors are informed with better private signals. We apply the model to study investors’ private word-of-mouth communications. The analysis indicates a “dark side” of word-of-mouth communications even when they are assumed to be unbiased and truthful. We generate empirical predictions regarding how market depth and firms’ disclosure qualities would change as technological innovations, such as social media, facilitate investors’ private communications.

**Keywords:** Public Disclosure; Private Word of Mouth; Price Volatility

**JEL Classifications:** D82; G14; M41

# 1 Introduction

Technological innovations, such as social media, have facilitated investors' private interpersonal communications and greatly changed the information environment in which firms operate. Shiller (2015, p.180) writes: "Word-of-mouth transmission of ideas appears to be an important contributor to day-to-day or hour-to-hour stock market fluctuations." As investors learn from both public disclosures and their private information channels, we ask two questions in this paper. First, how does the quality of investors' private signals affect firms' information environment (specifically, price volatility), and is the effect different from that resulting from firms' public disclosures? Second, how would a firm adjust its public disclosures in response to better informed investors? In particular, will investors' private information channels crowd out the firm's disclosures, or incentivize the firm to provide more precise public disclosures?

To answer these questions, we model an equilibrium asset market, a continuum of risk-averse investors, and a manager who operates the firm and chooses the precision of the public disclosure. The risk-averse manager (he) chooses an unobservable effort and an observable disclosure precision, and then sells his shares in a competitive market similar to Hellwig (1980) and Diamond and Verrecchia (1981). In addition to the firm's public disclosure, each of the continuum of investors (she) also observes an idiosyncratic private signal before trading.

We first analyze the similarities and differences between the two information channels in terms of their impact on price volatility. From an ex ante perspective, price volatility comes from two sources: (1) fundamental-driven volatility, attributed to the uncertainty of the underlying firm value, and (2) noise-driven volatility, attributed to noises that are unrelated to the firm value. Both private and public information increase fundamental-driven volatility by shifting uncertainties about the firm value from ex post to ex ante. The similarity between the two information channels echoes the well-known result that giving investors more information reduces the ex post uncertainty but increases the uncertainty ex ante (i.e., before the information is revealed).[1]

Interestingly, public disclosure and investors' private information can have an opposite effect on the noise-driven volatility. In particular, we show that while public disclosure unambiguously mitigates the impact of noisy supplies on price volatility, more precise private signals often amplify such supply shocks and, therefore, drive the price further away from its fundamental ex post. The key to understanding the result is investors' attempts to learn others' private information from the market price. As investors' private information becomes more precise, the equilibrium price aggregates the information dispersed among the investors more effectively. Anticipating a more informative price, each investor will optimally place a higher weight on the observed market price in valuing the firm. Ironically, when the investors rely more heavily on the market price in forming their beliefs, noises in the pricing process will too be amplified. In contrast, investors reduce their reliances on price when public disclosures become more precise. This is because improving disclosure quality does *not* change the information content of price and, therefore, investors will move their attention away from price and into the (more precise) public disclosure. As a result, improving disclosure quality always mitigates the impact of supply shocks on price.

[1]For example, Hirshleifer (1971, p. 568) states "the anticipation of public information becoming available in advance of trading adds a significant distributive risk to the underlying technological risk."This risk-shifting result is formally investigated in the cost of capital literature (e.g., Christensen et al., 2010; Dutta and Nezlobin, 2017; Gao, 2010).

We endogenize the manager's disclosure choice and examine how the optimal disclosure quality will change as the investors' private signals become more precise. Contrary to the casual intuition that investors' private information sources may crowd out firms' public disclosures, we show that the manager often chooses to provide more precise public disclosure in response to better privately informed investors. The thinking behind the complementarity is, as investors' private signals become more precise, price informativeness increases faster than the precision of private information per se. This motivates investors to place more weight on price in valuing the firm. Since an increase in investors' reliances on price has a side effect of amplifying noisy supplies in the pricing process, it indirectly increase the *marginal benefit* of public disclosure because more precise disclosure helps to mitigate such noise related price impact (and volatility). In particular, we show that better private information will incentivize the manager to improve public disclosure if and only if the variance of the noise supply is high. Intuitively, when supply shocks are volatile to begin with, the call for a more precise public disclosure to lower the otherwise exacerbated noise-driven volatility outweighs the intrinsic substitutability between the public and private signals in revealing information to the investors.

We apply our model to study investors' private word-of-mouth communications, using technology from the information percolation literature (e.g., Duffie and Manso, 2007; Duffie et al., 2009). We demonstrate that there is an endogenous "dark side" to investors' word-of-mouth communications: they amplify supply shocks and increase price volatility even when the communications are assumed to be unbiased, truthful, and informative. The results generate empirical predictions about which type of firm is more likely to increase or decrease its public disclosure quality in response to more active word-of-mouth communications. We show that, all else equal, a firm is more likely to increase its public disclosure quality following more active word of mouth if (1) investors are more risk averse, or (2) their private information endowment is less precise. We also show that more active word of mouth unambiguously lowers the market depth after endogenizing the firm's disclosure choice. These empirical predictions are relevant in light of the recent discussions about the consequences of the development of social media that facilitates interpersonal communications (e.g., Bartov et al., 2015; Blankespoor et al., 2014; Jung et al., 2018).

This paper is related to the literature on the relation between public and private information. Existing studies mostly focus on how anticipated public disclosure changes private incentives to acquire information and the implications on capital market (e.g., Diamond, 1985; Demski and Feltham, 1994; Kim and Verrecchia, 1994; McNichols and Trueman, 1994). Several papers show that releasing public information can crowd out private information acquisition by reducing the rents received by informed investors (e.g., Diamond, 1985; Fischer and Stocken, 2010; Gao and Liang, 2013; Han and Yang, 2013). Amador and Weill (2010) show a different crowding-out mechanism: more precise public information obscures the aggregation of agents' private information by making individuals' actions less sensitive to their private signals. Chen et al. (2014) show that, when investors have short horizons and are asymmetrically informed, public information can increase or decrease price informativeness. While the quality of public disclosures is generally taken as given in prior studies (for a review see Verrecchia, 2001; Goldstein and Yang, 2017), we examine how the firm revises its disclosure policy as investors' private information becomes more precise.[2] Our results show that better

[2]The voluntary disclosure literature (e.g., Dye, 1985; Verrecchia, 1983) focuses on the manager's ex-post information withholding decision. Stocken (2013) provides a comprehensive review of the

private information can actually incentivize more precise public disclosures once we take into account the information aggregation role of price.

Prior studies have shown informational complementarity in various settings. Kim and Verrecchia (1994) and Boot and Thakor (2001) show that public disclosure can strengthen investors' incentives to acquire private information if the two are complementary in understanding the fundamentals. Demski and Feltham (1994) and McNichols and Trueman (1994) show that public disclosure can stimulate private information acquisition in a setting where the investors trade on their acquired private information before the public announcement. Arya et al. (2017) demonstrate natural synergies between accounting reports and stock prices in directing firm strategies. Diamond and Verrecchia (1991) consider a setting in which only some investors have private information, and the firm increases its public disclosure to lower information asymmetry and hence its cost of capital. Goldstein and Yang (2015) show that investors' information acquisition can be complements if their information concerns different pieces of the fundamental value. Hellwig and Veldkamp (2009) show that if agents' actions are assumed to be strategic complements, then their information acquisitions are also strategic complements. Our mechanism does not require investors' incentives to coordinate, higher-order beliefs, or a division between informed and uninformed investors.

Angeletos and Werning (2006) study a currency attack type of coordination game and show that less noise may increase price volatility. This noise-amplifying result also arises in our model, but the two papers rely on entirely different mechanisms. Angeletos and Werning (2006, p.1722) write: "This novel coordinating role is crucial for our results on price multiplicity and price volatility." There is no incentive to coordinate

subject.

in our model. Instead, our result suggests that the potential cost of better private information in amplifying noise-driven (or non-fundamental) price volatility is a feature intrinsic to noisy rational expectation models.

The paper proceeds as follows. Section 2 describes the model. Section 3 takes the disclosure precision as exogenous and analyze the similarity and difference between public disclosure and private signals. Section 4 endogenizes the disclosure precision and derives the necessary and sufficient condition under which the firm improves its disclosure quality in response to better informed investors. Section 5 applies the model to study investors' private word-of-mouth communications and discusses empirical predictions. Section 6 concludes.

# 2 Model Setup

The model consists of a risk-averse manager who operates a firm and a continuum of risk-averse investors. At the beginning of the game, the manager chooses an unobservable effort $a \geq 0$ at a personal cost $C(a) = \frac{1}{2}a^2$. The manager's effort $a$ increases the firm's value $v$ in the following stochastic manner:

$$v = a + \phi, \tag{1}$$

where $\phi \sim N(0, \sigma_\phi^2)$ is normally distributed with mean zero and precision $\tau_\phi = 1/\sigma_\phi^2$.

Given a realization of the firm value $v$, the firm is traded in a competitive market in which the market-clearing price $p$ is determined. The manager owns an exogenous amount of shares that we normalize to one.[3] In choosing his unobservable effort $a$ at

[3]A literal interpretation is that the manager/entrepreneur initially owns one hundred percent of the firm and later sells the firm at $t = 2$. However, this normalization (hence the entrepreneur-IPO

$t = 0$, the manager maximizes his expected constant absolute risk aversion (CARA) utility as follows:

$$U^M = \mathbb{E}\left[-\exp\left(-\rho\left(p - C(a)\right)\right)\right], \tag{2}$$

where $\rho$ is the manager's constant absolute risk aversion, $p$ is the equilibrium price at which he sells his shares, and $C(a) = a^2/2$ is his personal cost of effort.

The price $p$ is determined in a competitive market similar to Hellwig (1980) and Diamond and Verrecchia (1981). There is a continuum of investors $i \in [0, 1]$ and a risk-free asset that serves as the numeraire. Noisy traders provide liquidity in the sense that they supply $\varepsilon$ units of the firm's share per capita to the market, and we assume $\varepsilon \sim N(0, \sigma_\varepsilon^2)$. Each investor is endowed with $w_0$ units of the risk-free asset and has the same exponential utility function:

$$U_i = -\exp(-W_i/r), \tag{3}$$

where $W_i$ is investor $i$'s ending wealth and $r$ is the common risk tolerance.

Prior to the trading stage, the firm publicly discloses a signal $x$ that is informative about the firm's value:

$$x = v + \zeta, \tag{4}$$

with $\zeta \sim N(0, \sigma_x^2)$. The precision of the public disclosure, $\tau_x = 1/\sigma_x^2$, is publicly chosen by the manager at $t = 0$. The disclosure choice $\tau_x$, as argued in Diamond and Verrecchia (1991) and Kanodia and Lee (1998), can be interpreted as the choice of an accounting technique or a committed policy of providing earnings guidance or other forecasts.[4]

interpretation) is not important: we can assume that the manager owns $\alpha < 1$ fraction of the firm and verify that our results carry over qualitatively.

[4]This assumption is standard in the literature. See, for example, Admati and Pfleiderer (2000);

In addition to the the firm's public disclosure, each investor $i \in [0,1]$ receives a private signal $y_i$ about $v$ prior to her trading and we assume

$$y_i = v + \eta_i, \tag{5}$$

where $\eta_i \sim N(0, \sigma_\eta^2)$ is independent across all investors and their signal precision $\tau_\eta = 1/\sigma_\eta^2$ is the same across all investors. Figure 1 summarizes the sequence of the game.

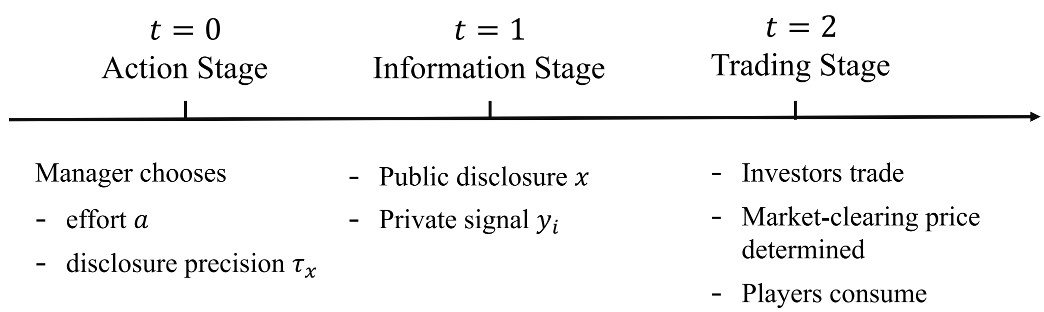


Figure 1: Time line

# 3 Analysis with Exogenous Disclosure Precision

In this section, we take the precision of the public disclosure as given and solve for the manager's equilibrium effort and the subsequent trading game. We also demonstrate how firm's public disclosure and investors' private signals can have qualitatively different effects on investors' inferences from price and price volatility.

Fishman and Hagerty (1989); Kurlat and Veldkamp (2015).

## 3.1 Equilibrium

Our trading subgame is built on Diamond and Verrecchia (1981) and incorporates a public disclosure as well as an unobservable effort. The equilibrium is solved in three steps. We first reason from investors' perspective and solve for the linear pricing function that clears the market, while taking investors' *conjecture* $\hat{a}$ about the manager's effort as given. In particular, we guess and verify the following linear pricing function:

$$p(\hat{a}) = \hat{\alpha}_0 + \hat{\alpha}_v v + \hat{\alpha}_x \zeta - \hat{\alpha}_\varepsilon \varepsilon, \tag{6}$$

where the coefficients can depend on the conjectured effort $\hat{a}$ but not the actual $a$ that is unobservable by assumption.

In the second step, we reason from the manager's perspective. The manager, taking the market conjecture $\hat{a}$ and the pricing function (6) as given, chooses $a$ to maximize his payoff (2). Given the CARA-normal setup, this is equivalent to maximizing the following certainty equivalent:

$$\max_a \mathbb{E}[p|a, \hat{a}, \tau_x] - C(a) - \frac{\rho}{2}\mathrm{var}(p|a, \hat{a}, \tau_x),$$

where $\mathbb{E}[p|a, \hat{a}] = \hat{a} + \hat{\alpha}_v a$ and $\mathrm{var}(p|a, \hat{a})$ are derived from (6). The first-order condition yields the manager's best response:

$$a^*(\hat{a}, \tau_x) = \hat{\alpha}_v. \tag{7}$$

In the third step, we impose rational expectations to determine the equilibrium. That is, the conjectured effort equals the actual one in equilibrium (i.e., $a^* = \hat{a}$) and the conjectured linear pricing function coincides with the actual market-clearing price. We summarize the equilibrium in Proposition 1 and defer the details to the Appendix.[5]

**Proposition 1 (Equilibrium with Exogenous Precision)** *Fixing the disclosure precision* $\tau_x$*, there exists a unique linear pricing function:*

$$p = \alpha_0 + \alpha_v v + \alpha_x \zeta - \alpha_\varepsilon \varepsilon, \tag{8}$$

*where* $\alpha_0 = \frac{\tau_\phi}{\tau_\phi+\tau_x+\tau_\eta+\tau_p} a^*, \alpha_v = \frac{\tau_x+\tau_\eta+\tau_p}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}, \alpha_x = \frac{\tau_x}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}, \alpha_\varepsilon = \frac{1}{r\tau_\eta}\frac{\tau_\eta+\tau_p}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}$*, and* $\tau_p = \left(\tau_\eta r\right)^2 \tau_\varepsilon$ *is the precision of price used as an independent signal of* $v$*. The manager's equilibrium effort choice is*

$$a^* = \alpha_v = 1 - \tau_\phi \mathrm{var}(v|\mathcal{F}), \tag{9}$$

*where* $\mathrm{var}(v|\mathcal{F}) = \frac{1}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}$ *is an individual investor's residual uncertainty about* $v$ *given her information set* $\mathcal{F} = \{x, y_i, p\}$*.*

**Proof.** All proofs are in the Appendix. ■

The pricing function (8) suggests that, all else equal, the market-clearing price $p$ will be higher if the firm's fundamental $v$ is higher, the asset supply $\varepsilon$ is lower, or the common noise $\zeta$ contained in the public disclosure $x$ is higher. The idiosyncratic noises $\eta_i$ contained in investors' private signals $y_i$ do not affect the price because they are aggregated away by the law of large numbers. In equilibrium, observing price $p$ is

[5] We show in the proof that the coefficient $\hat{\alpha}_v$ in (6) is independent of the investor's conjecture $\hat{a}$ and only depends on the primitives in the model that are commonly known. Therefore, equation (7) suggests that the manager has a dominant strategy $a^*$ in the sense that it is independent of the market's belief $\hat{a}$. Such a dominant strategy response rules out potential multiple equilibria. We thank Phillip Stocken for helping us with the uniqueness argument.

informationally equivalent to observing

$$q \doteq (p - \alpha_x x - \alpha_0)/(\alpha_v - \alpha_x) = v - \frac{\epsilon}{r\tau_\eta}, \tag{10}$$

which is a normally distributed signal of the firm value $v$ with a precision $\tau_p = (\tau_\eta r)^2 \tau_\varepsilon$.

To understand the manager's equilibrium effort choice $a^* = \alpha_v$, note that while the market price satisfies $\mathbf{E}[p] = \mathbf{E}[v]$ in equilibrium, it is formed in a process that is only partially responsive to its fundamental value $v$ (and hence effort $a$) in the sense that $\frac{d}{dv}\mathbf{E}[p|v] = \alpha_v < 1$. This partial responsiveness arises because, when assessing the firm value, investors always attach some weight to the conjectured effort level $\hat{a}$ that the manager takes as given and cannot change (see Holmström and Tirole, 1993; Edmans and Manso, 2011, for a similar argument). The manager's moral hazard problem arises because the rate at which his effort increases the market price $\frac{d\mathbf{E}[p|a]}{da} = \alpha_v$ is lower than the rate at which it increases the firm's value $\frac{d\mathbf{E}[v|a]}{da} = 1$. The coefficient $\alpha_v$ measures the manager's perceived marginal benefit of exerting effort, which explains his effort choice $a^* = \alpha_v$ in equilibrium.

Given the manager's equilibrium effort $a^* = 1 - \frac{\tau_\phi}{\tau_\phi + \tau_x + \tau_\eta + \tau_p}$, it is clear that improving the precision of either public or private information (i.e., $\tau_x$ or $\tau_\eta$) will incentivize the manager to exert more effort. This, however, does not mean that the manager always benefits from a more precise information environment. The reason is that revealing information to investors prior to trading can also make the price $p$ more volatile from an ex ante point of view, and the higher price volatility lowers the manager's certainty equivalent $CE = \mathbb{E}[p|a^*] - C(a^*) - \frac{\rho}{2}\mathrm{var}(p)$.

### 3.2 Inferences from Price and Noise Amplification

To analyze how the two types of information affect price volatility $\text{var}(p)$ (and, hence the manager's certainty equivalent), we decompose $\text{var}(p)$ as

$$\text{var}(p) = \underbrace{\text{var}\left[\mathbb{E}(p|v)\right]}_{\text{Fundamental-driven volatility: } V_F} + \underbrace{\mathbb{E}\left[\text{var}\left(p|v\right)\right]}_{\text{Noise-driven volatility: } V_N}. \tag{11}$$

The Fundamental-driven volatility $V_F \doteq \text{var}\left[\mathbb{E}(p|v)\right] = \alpha_v^2\sigma_\phi^2$ is caused by the volatility of the underlying firm value $v$, and the Noise-driven volatility $V_N \doteq \mathbb{E}\left[\text{var}\left(p|v\right)\right] = \alpha_x^2\sigma_x^2 + \alpha_\varepsilon^2\sigma_\varepsilon^2$ captures the volatility driven by noises unrelated to the fundamental $v$.

An immediate observation is that fundamental-driven volatility $V_F$ will be higher if we improve the precision of either public disclosure or investors' private signals. The intuition can be illustrated by rewriting $V_F$ as follows:

$$V_F \doteq \text{var}\left[\mathbb{E}(p|v)\right] = \alpha_v^2\sigma_\phi^2 = \left(1 - \tau_\phi \text{var}(v|\mathcal{F})\right)^2 \times \sigma_\phi^2. \tag{12}$$

It follows from (12) that any reduction of ex-post uncertainty $\text{var}(v|\mathcal{F})$ due to new information – be it public or private – is accompanied by an increase in ex-ante fundamental-driven volatility $V_F$. That is, releasing new information prior to trading redistributes the uncertainty from ex post to ex ante. This risk-shifting result dates back to Hirshleifer (1971) and Ross (1989), and is formally investigated in the cost of capital literature (e.g., Christensen et al., 2010; Dutta and Nezlobin, 2017; Gao, 2010).

While both public and private information increase the fundamental-driven volatility $V_F$, they affect the noise-drive volatility $V_N = \alpha_x^2\sigma_x^2 + \alpha_\varepsilon^2\sigma_\varepsilon^2$ differently. In particular, the two types of information have qualitatively different effects on the volatility introduced by noisy supply shocks, i.e., $\alpha_\varepsilon^2\sigma_\varepsilon^2$. We focus our discussions on $\alpha_\varepsilon^2\sigma_\varepsilon^2$ part of the noise-drive volatility because it is the key to understanding why the manager may proactively improve disclosure quality in response to better privately informed investors in Section 4.

Investors' inferences from market price are important in understanding the differences between public and private information. To illustrate, we rewrite the pricing function (8) in Proposition 1 as

$$p = \int_i \mathbb{E}\left(v|\mathcal{F}_i\right) di - \frac{\mathrm{var}\left(v|\mathcal{F}\right)}{r}\varepsilon, \tag{13}$$

where $\mathcal{F}_i = \{\hat{a}, x, y_i, p\}$ is investor $i$'s information set. That is, the market-clearing price equals the aggregate belief among investors minus a risk premium that investors demand to absorb the liquidity shock $\varepsilon$. It follows from Bayes' rule that $i$'s posterior assessment $\mathbb{E}\left(v|\mathcal{F}_i\right)$ is a precision-weighted average of signals in her information set:

$$\mathbb{E}\left(v|\mathcal{F}_i\right) = w_0\hat{a} + w_x x + w_y y_i + w_q q, \tag{14}$$

where $w_0 = \frac{\tau_\phi}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}, w_y = \frac{\tau_\eta}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}, w_q = \frac{\tau_p}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}, w_x = \frac{\tau_x}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}$, and $q = v - \frac{\epsilon}{r\tau_\eta}$ is informationally equivalent to price $p$ and defined in (10). Substituting $\mathbb{E}\left(v|\mathcal{F}_i\right)$ into (13) and using the fact $\int_i y_i d_i = v$, we simplify (13) to be $p = \left[w_0\hat{a} + w_x x + w_y v + w_q(v - \frac{\epsilon}{r\tau_\eta})\right] - \frac{\mathrm{var}\left(v|\mathcal{F}\right)}{r}\varepsilon$, which allows us to show

$$\begin{aligned} \alpha_\epsilon = |\frac{dp}{d\varepsilon}| =& \frac{\mathrm{var}\left(v|\mathcal{F}\right)}{r} + |\frac{d}{d\varepsilon}\int_i \mathbb{E}\left(v|\mathcal{F}_i\right) di\,| \\ =& \frac{\mathrm{var}\left(v|\mathcal{F}\right)}{r} + \frac{w_q}{r\tau_\eta}. \end{aligned} \tag{15}$$

Equation (15) shows that a supply shock $\varepsilon$ moves the market price in two ways. First, it affects the risk premium $\frac{\mathrm{var}\left(v|\mathcal{F}\right)}{r}$ required to compensate investors for clearing the supply shock. Intuitively, investors demand a higher premium when they face more uncertainties (i.e., higher $\mathrm{var}(v|\mathcal{F})$) or are less tolerant to risk (i.e., lower $r$). Second, $\varepsilon$ affects investors' aggregate assessment of the firm value $\int_i \mathbb{E}\left(v|\mathcal{F}_i\right) di$ due to their inferences from price, i.e., $w_q > 0$. The second channel arises because when investors use price to infer firm's value, they cannot tell whether a price movement is caused by a change in asset *supply* $\varepsilon$ or a change in investors' *demand*. As a result, a low price caused by a positive supply shock $\varepsilon$ will be partially interpreted by each investor as others receiving unfavorable signals, pushing down the aggregate belief $\int_i \mathbb{E}\left(v|\mathcal{F}_i\right) di$. It is this imperfect inferences that investors draw from price that endogenously amplify the price impact of a random supply shock. The following result shows that public and private information affect investors' inferences from price differently.

**Lemma 1 (Different Effects on Inferences)** *Investors rely more on $p$ when their private signals are more precise. In contrast, they rely less on $p$ when the public disclosure becomes more precise. That is,*

$$\frac{d}{d\tau_\eta} w_q > 0, \quad \frac{d}{d\tau_x} w_q < 0, \tag{16}$$

*where $w_q = \frac{\tau_p}{\tau_\phi + \tau_x + \tau_\eta + \tau_p}$ is the weight investors place on price $p$ in assessing the firm value and is defined in* (14).

The thinking behind the lemma can be illustrated by examining investors $i$'s demand function

$$D_i = \frac{r\left[\mathbb{E}\left(v|\mathcal{F}_i\right) - p\right]}{\mathrm{var}\left(v|\mathcal{F}_i\right)} = r\,\tau_\eta(y_i - v) + \varepsilon. \tag{17}$$

As private information becomes more precise (i.e., higher $\tau_\eta$), investors trade more aggressively on the private signal $y_i$. As a result, the market price better aggregates investors' idiosyncratic private information and, hence, becomes a more informative signal about the firm value (i.e., higher $\tau_p$). Note that price informativeness $\tau_p$ grows *faster* than the precision of private signal so that the ratio $\frac{\tau_p}{\tau_\eta} = r^2 \tau_\eta \tau_\varepsilon$ increases in $\tau_\eta$. Anticipating an increasingly informative $\tau_p$ (relative to private signals), investors optimally increase the weight $w_q$ given to the signal in forming their posterior beliefs, which explains $\frac{d}{d\tau_\eta} w_q > 0$. In contrast, a similar increase in reliance on price does *not* occur when we improve public disclosure precision $\tau_x$. This is because public disclosures are observed by everyone and will not affect investors' trading strategy (17) and, hence, price informativeness $\tau_p$. Without affecting price informativeness, we know from Bayes' rule that a higher $\tau_x$ induces investors to rely more on the public disclosure $x$ and lowers the weights given to other signals in the information set $\mathcal{F}_i$.

Our next result follows from Lemma 1 and provides another aspect to compare the different effects public and private information have on firm's information environment.

**Proposition 2 (Supply Shock Mitigation versus Amplification)** *More precise public disclosure mitigates the impact of the noisy supply $\varepsilon$ on price: $\frac{d}{d\tau_x}\alpha_\varepsilon < 0$. In comparison, more precise private information amplifies $\alpha_\varepsilon$ when $\tau_\varepsilon \in (\underline{\tau}_\varepsilon, \bar{\tau}_\varepsilon)$.*[6]

To see why improving disclosure quality $\tau_x$ unambiguously lowers $\alpha_\varepsilon = \frac{\text{var}(v|\mathcal{F})}{r} + \frac{w_q}{r\tau_\eta}$, note that a higher $\tau_x$ reduces both the risk premium $\text{var}(v|\mathcal{F})/r$ and the sensitivity of aggregate belief to supply shock $\left|\frac{d}{d\varepsilon}\int_i \mathbb{E}\left(v|\mathcal{F}_i\right) di\right| = \frac{w_q}{r\tau_\eta}$ (recall $\frac{dw_q}{d\tau_x} < 0$). Improving private signal precision $\tau_\eta$ has a different effect. In particular, while also reducing the risk-premium $\text{var}(v|\mathcal{F})/r$, a higher $\tau_\eta$ can increases $\alpha_\varepsilon$ because it motivates investors

[6] We specify the exogenous boundaries $\underline{\tau}_\varepsilon$ and $\bar{\tau}_\varepsilon$ in the Appendix.

to rely more on the price (recall $\frac{dw_q}{d\tau_\eta} > 0$). Proposition 2 shows that the effect of higher $\tau_\eta$ in inducing more reliance on price dominates (and, hence, $\frac{d\alpha_\varepsilon}{d\tau_\eta} > 0$) as long as the supply noise is not too extreme. The intuition can be illustrated by analyzing the limiting case of $\tau_\varepsilon \to 0$ or $\tau_\varepsilon \to \infty$: price $p$ will be either completely uninformative (for $\tau_\varepsilon \to 0$) or perfectly informative (for $\tau_\varepsilon \to \infty$). In both cases, however, the marginal effect of private information $\tau_\eta$ on investors' inference $w_q$ diminishes (i.e., $\frac{dw_q}{d\tau_\eta} \to 0$). Without a meaningful influence on the investors' inferences from price $w_q$, it follows from (15) that providing more precise information will lower the price impact of the noisy supply.

# 4 Endogenous Disclosure and Complementarity

In this section, we endogenize the optimal precision choice $\tau_x^*$ and investigate how a firm would adjust its disclosure precision $\tau_x^*$ in response to better informed investors. Contrary to the casual intuition that investors' private information sources may crowd out firms' public disclosures, we show that firms often choose to provide more precise public disclosure when investors' private information becomes more precise.

## 4.1 Optimal Disclosure Quality

The manager takes the pricing function $p$ and his equilibrium effort $a^*$ in Proposition 1 as given and chooses an optimal disclosure quality $\tau_x^*$ to maximize his certainty equivalent $\mathbb{E}[p|a^*]-C(a^*)-\frac{\rho}{2}\text{var}(p)$, which can be simplified as follows after substituting the pricing function from Proposition 1 and $\text{var}(p) = V_F + V_N$ from (11):

$$CE = a^* - \frac{(a^*)^2}{2} - \frac{\rho}{2}(V_F + V_N). \tag{18}$$

Differentiating the manager's certainty equivalence above, we obtain the following first-order condition (FOC) of $\tau_x$:

$$FOC = \underbrace{\tau_\phi^2 \text{var}^3(v|\mathcal{F})}_{MB \text{ on } \mathbb{E}[p|a^*]-C(a^*)} \quad \underbrace{-\rho\left(1-\tau_\phi \text{var}\left(v|\mathcal{F}\right)\right)\text{var}^2(v|\mathcal{F})}_{MC \text{ on } V_F}$$
$$\underbrace{+\rho\overbrace{\left(\tau_x \text{var}\left(v|\mathcal{F}\right)-\frac{1}{2}\right)\text{var}^2(v|\mathcal{F})}^{-\frac{\rho}{2}\frac{d(\alpha_x^2\sigma_x^2)}{d\tau_x}}+\rho\overbrace{\frac{\left(1+\frac{\tau_p}{\tau_\eta}\right)^2}{r^2\tau_\varepsilon}\text{var}^3(v|\mathcal{F})}^{-\frac{\rho}{2}\frac{d(\alpha_\varepsilon^2\sigma_\varepsilon^2)}{d\tau_x}}}_{\text{Marginal Effect on } V_N} = 0. \tag{19}$$

The first two terms of the FOC above summarize the previously discussed marginal benefit of improving $\tau_x$ in motivating a higher equilibrium effort (net of its cost) and the marginal cost in driving up the fundamental-driven volatility $V_F$, respectively.[7] The negative sign in front of $\rho$ reflects the fact that higher volatilities reduce the manager's certainty equivalence. The third term of FOC (19) is the marginal effect of disclosure quality $\tau_x$ on the noise-driven volatility $V_N$. It is worth noting that improving disclosure quality $\tau_x$ affects the two components of the price volatility differently: while a higher $\tau_x$ always increases fundament-driven volatility $V_F$, it can actually lower the noise-driven volatility $V_N$. In particular, Proposition 2 shows that a higher $\tau_x$ reduces supply-shock related volatility $\alpha_\varepsilon^2\sigma_\varepsilon^2$, suggesting an additional a marginal benefit of $\tau_x$.

Solving the first-order condition (19) yields a unique optimal precision $\tau_x^*$. We summarize the equilibrium in Proposition 3.

**Proposition 3 (Equilibrium)** *When the precision of public disclosure is endogenous,*

[7] We use the fact $\frac{\partial \text{var}\left(v|\mathcal{F}\right)}{\partial \tau_x} = -\text{var}^2\left(v|\mathcal{F}\right)$ in the derivation.

*the game has a unique linear equilibrium in which the disclosure precision is*

$$\tau_x^* = \max\{0,\, 2\frac{1}{r^2\tau_\varepsilon} + \frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \tau_\eta - \left(\tau_\eta r\right)^2 \tau_\varepsilon\}, \tag{20}$$

*with* $\tau_x^* > 0$ *if and only if* $\sigma_\varepsilon^2 > \frac{r^2\left[\sqrt{\left(\frac{2}{\rho}\tau_\phi^2-\tau_\phi+\tau_\eta\right)^2+8\tau_\eta^2}-\left(\frac{2}{\rho}\tau_\phi^2-\tau_\phi+\tau_\eta\right)\right]}{4}$. *Substituting the value of* $\tau_x^*$ *into Proposition 1 fully characterizes the equilibrium.*

## 4.2 Response to Changes in Private Information

How would the manager adjust the optimal disclosure precision $\tau_x^*$ in response to better privately informed investors? We are particularly interested in showing when and why more precise private signals can motivate the manager to provide more precise public disclosure as well. We apply implicit function theorem to FOC (19) to obtain

$$\frac{d\tau_x^*}{d\tau_\eta} = -\frac{dFOC}{d\tau_\eta} / \frac{dFOC}{d\tau_x}, \tag{21}$$

where the denominator $\frac{dFOC}{d\tau_x}$ is the second-order condition of the manager's maximization problem and satisfies $\frac{dFOC}{d\tau_x}|_{\tau_x=\tau_x^*} = -\frac{\rho}{2}\text{var}^3(v|\mathcal{F}) < 0$. Therefore,

$$\frac{d\tau_x^*}{d\tau_\eta} \propto \frac{dFOC}{d\tau_\eta} = \underbrace{\frac{\partial FOC}{\partial \text{var}\left(v|\mathcal{F}\right)}\frac{\partial \text{var}\left(v|\mathcal{F}\right)}{\partial \tau_\eta}}_{\text{Information Revelation Effect}} + \underbrace{\frac{\partial FOC}{\partial \frac{\tau_p}{\tau_\eta}}\frac{\partial \frac{\tau_p}{\tau_\eta}}{\partial \tau_\eta}}_{\text{Inference Effect}}. \tag{22}$$

The equation above suggests that the way investors' private information $\tau_\eta$ changes the manager's $FOC$ (and, hence, his disclosure choice $\tau_x^*$) can be summarized into two effects. First, more precise private signals reveal more information about the firm value and, therefore, reduce uncertainties $\text{var}(v|\mathcal{F})$ investors face in equilibrium.

Second, a higher $\tau_\eta$ also improves the information content of market price *relative to* that of investors' private information per se (i.e., a higher precision ratio $\frac{\tau_p}{\tau_\eta}$), which incentivizes investors to rely more on price in making inferences. We use "Inference Effect" to label the second effect (of $\tau_\eta$) that works through changing $\frac{\tau_p}{\tau_\eta}$ in (22). To see the link between the ratio $\frac{\tau_p}{\tau_\eta}$ and investors' inferences from price, note

$$\frac{\tau_p}{\tau_\eta} = r^2 \tau_\eta \tau_\varepsilon = \frac{w_q}{w_y}, \tag{23}$$

where $w_q$ (and $w_y$) are the weights each investor places on market price $p$ (and private signal $y$, respectively) suggested by Bayes' rule (14). That is, the precision ratio $\frac{\tau_p}{\tau_\eta}$ captures investors' reliance on price relative to their private information, i.e., $\frac{w_q}{w_y}$.

Further examination of the manager's $FOC$ suggests that the "Inference Effect" in (22) comes solely from the marginal benefit of public disclosures in reducing $\alpha_\varepsilon^2 \sigma_\varepsilon^2$ part of the volatility, i.e., $\frac{d(-\frac{\rho}{2}\alpha_\varepsilon^2\sigma_\varepsilon^2)}{d\tau_x}$. We can then write the Inference Effect as

$$\frac{\partial FOC}{\partial \frac{\tau_p}{\tau_\eta}} \frac{\partial \frac{\tau_p}{\tau_\eta}}{\partial \tau_\eta} = \frac{\partial \left( \frac{d(-\frac{\rho}{2}\alpha_\varepsilon^2\sigma_\varepsilon^2)}{d\tau_x} \right)}{\partial \frac{\tau_p}{\tau_\eta}} \frac{\partial \frac{\tau_p}{\tau_\eta}}{\partial \tau_\eta} = 2\rho \mathrm{var}^3(v|\mathcal{F}) \left( 1 + \frac{\tau_p}{\tau_\eta} \right). \tag{24}$$

Substituting (24) back to (22), we fully characterize the relation $\frac{d\tau_x^*}{d\tau_\eta}$ as follows[8]

$$\begin{aligned} \frac{d\tau_x^*}{d\tau_\eta} = -\frac{\frac{dFOC}{d\tau_\eta}}{\frac{dFOC}{d\tau_x}} &= \left( \frac{\partial FOC}{\partial \mathrm{var}\left(v|\mathcal{F}\right)} \frac{\partial \mathrm{var}\left(v|\mathcal{F}\right)}{\partial \tau_\eta} + \frac{\partial FOC}{\partial \frac{\tau_p}{\tau_\eta}} \frac{\partial \frac{\tau_p}{\tau_\eta}}{\partial \tau_\eta} \right) \frac{1}{\frac{\rho}{2}\mathrm{var}^3(v|\mathcal{F})}, \\ &= \underbrace{-3(1+2\frac{\tau_p}{\tau_\eta})}_{\text{Information Revelation Effect (-)}} + \underbrace{4(1+\frac{\tau_p}{\tau_\eta})}_{\text{Inference Effect (+)}} . \end{aligned} \tag{25}$$

[8]Recall $\frac{dFOC}{d\tau_x}|_{\tau_x=\tau_x^*} = -\frac{\rho}{2}\mathrm{var}^3(v|\mathcal{F})$, and we show $\frac{\partial FOC}{\partial \mathrm{var}(v|\mathcal{F})}|_{\tau_x=\tau_x^*} = \frac{3\rho}{2}\mathrm{var}(v|\mathcal{F})$ in the appendix.

A noticeable feature of (25) is that the two channels we identify separates the countervailing effects that private information has on the optimal disclosure choice $\tau_x^*$. What motivates a potential increase in disclosure quality is the "Inference Effect", which is rooted in $\alpha_\varepsilon^2\sigma_\varepsilon^2$ part of the noise-driven volatility $V_N$ that better disclosure helps mitigate. Note that the marginal benefit of a higher disclosure quality $\tau_x$ in lowering $\alpha_\varepsilon^2\sigma_\varepsilon^2$ is shown in (19) to be $\frac{(1+\frac{\tau_p}{\tau_\eta})^2}{r^2\tau_\varepsilon}\text{var}^3(v|\mathcal{F})$, and, therefore, better private signals (i.e, higher $\tau_\eta$) can potentially strengthen the marginal benefit by increasing the ratio $\frac{\tau_p}{\tau_\eta} = r^2\tau_\eta\tau_\varepsilon$.[9] Intuitively, as investors' private signals become more precise, price informativeness increases faster than investors' private information per-se (i.e., higher $\frac{\tau_p}{\tau_\eta}$). This incentivizes investors to rely increasingly more on price in valuing the firm. Investors' increased reliances on price endogenously amplifies the liquidity shock $\varepsilon$ and, therefore, indirectly increase the marginal benefit of better public disclosure because more precise disclosure helps to mitigate this type of noise-driven volatility. The increased marginal benefit of $\tau_x$ in lowering $\alpha_\varepsilon^2\sigma_\varepsilon^2$ is formally shown in (24) and underlies the positive "Inference Effect" in (25).

The unambiguous negative "Information Revelation Effect" in (25) suggests that public and private information play a qualitatively similar role (and, hence, are substitutive to each other) if what matters is the equilibrium informativeness per se, captured by var $\left(v|\mathcal{F}\right)$. We illustrate the thinking by showing that, if we only focus on its effect in lowering $\text{var}(v|\mathcal{F})$, an increase in $\tau_\eta$ weakens the marginal benefit of improving disclosure quality $\tau_x$ relative to its marginal cost. To do so, we normalize

[9] A higher $\tau_\eta$ also reduces var $\left(v|\mathcal{F}\right)$, which tends to lower the marginal benefit. This countervailing effect is captured by Information Revelation Effect in (25).

the manager's first-order condition ($FOC$) as follows:

$$foc \doteq \frac{FOC}{\mathrm{var}^2(v|\mathcal{F})} = \underbrace{\tau_\phi^2 \mathrm{var}(v|\mathcal{F})}_{MB \text{ on Effort}} \underbrace{- \rho\left(1 - \tau_\phi \mathrm{var}\left(v|\mathcal{F}\right)\right)}_{MC \text{ on } V_F}$$
$$\underbrace{+\rho\left(\tau_x \mathrm{var}\left(v|\mathcal{F}\right) - \frac{1}{2}\right) + \rho\frac{\left(1+\frac{\tau_p}{\tau_\eta}\right)^2}{r^2\tau_\varepsilon}\mathrm{var}(v|\mathcal{F})}_{\text{Marginal Effect on } V_N} = 0. \tag{26}$$

The normalized $foc$ above captures the *relative* marginal benefit and cost of improving disclosure quality $\tau_x$ as it scales all the marginal effects in the original $FOC$ by $1/\mathrm{var}^2(v|\mathcal{F})$.[10] Inspecting the normalized $foc$ (26), we notice that, by lowering $\mathrm{var}(v|\mathcal{F})$, more precise private information (higher $\tau_\eta$) reduces the relative marginal benefit of public disclosure (e.g., MB on effort), while at the same time, increases the relative marginal cost of disclosure (e.g., MC on $V_F$). Both effects suggest that a higher $\tau_\eta$ would unambiguously reduce the optimal disclosure choice $\tau_x^*$ if we only cared about the role of private information $\tau_\eta$ in reducing uncertainty $\mathrm{var}(v|\mathcal{F})$,

The net effect of the two countervailing forces shown in (25) determines whether the manager increases or decreases the optimal disclosure quality $\tau_x^*$ in response to better privately informed investors. Proposition 4 shows that the complementarity embedded in the Inference Effect in (25) dominants the substitutive effect when the noisy supply is expected to be volatile.

**Proposition 4 (Stimulating More Disclosure)** *As investors' private signals become more precise, the manager responds by improving the firm's disclosure quality*

[10]Note that (26) is mathematically equivalent to the original $FOC$ (19) in the sense that $FOC = 0$ if and only if $foc = 0$.

*if the variance of the noisy supply is large. That is,*

$$\frac{d}{d\tau_\eta}\tau_x^* \geq 0 \text{ \textit{if and only if} } \sigma_\varepsilon^2 > 2\tau_\eta r^2, \tag{27}$$

*and the above inequality is strict (i.e.,* $\frac{d}{d\tau_\eta}\tau_x^* > 0$*) for any* $\tau_x^* > 0$*.*

To gain some intuition about the condition $\sigma_\varepsilon^2 > 2\tau_\eta r^2$, first note from (25) that both the (substitutive) Information Revelation Effect and the (complementary) Inference Effect are stronger for a higher $\frac{\tau_p}{\tau_\eta}$. The fact that Inference Effect increases in $\frac{\tau_p}{\tau_\eta}$ is intuitive. Recall the side effect of investors' inferences from price (in terms of amplifying $\alpha_\varepsilon^2\sigma_\varepsilon^2$) is what motivates the manager to improve disclosure quality $\tau_x$ in response to better privately informed investors. We therefore expect such incentives to be stronger when investors' relative reliance on price (and, hence, its side effect) is high to begin with, which occurs when $\frac{\tau_p}{\tau_\eta}$ is high. To see why the Information Revelation Effect – through $\text{var}(v|\mathcal{F})$ – is also stronger for higher $\frac{\tau_p}{\tau_\eta}$, we use the following equation to capture the comparative advantage of private information precision $\tau_\eta$ (over that of public disclosures) in reducing investors' equilibrium uncertainties $\text{var}(v|\mathcal{F})$:

$$\frac{d\text{var}(v|\mathcal{F})}{d\tau_\eta} / \frac{d\text{var}(v|\mathcal{F})}{d\tau_x} = 1 + 2\frac{\tau_p}{\tau_\eta}. \tag{28}$$

The comparative advantage comes from the information aggregation role of market price and is captured by $2\frac{\tau_p}{\tau_\eta}$ in (28). For higher $\frac{\tau_p}{\tau_\eta}$, private information are increasingly more efficient than public disclosures in revealing information to the market, reducing the need to improve disclosure quality $\tau_x$ for the purpose of resolving investors' uncertainties. Moreover, condition (25) further shows that a higher $\frac{\tau_p}{\tau_\eta}$ increases the Information Revelation Effect faster than it increases the Inference Effect. Therefore, the Inference Effect dominates (and, hence, $\frac{d\tau_x^*}{d\tau_\eta} > 0$) when $\frac{\tau_p}{\tau_\eta} = r^2\tau_\eta\tau_\varepsilon$ is not too high. That is, the information aggregation role of market price cannot be too efficient, which occurs when the noisy supply is volatile enough, i.e., $\sigma_\varepsilon^2 = \frac{1}{\tau_\varepsilon}$ is large enough. Intuitively, when the random shock $\varepsilon$ is volatile to start with, the call for a more precise public disclosure to lower the otherwise exacerbated noise-driven volatility outweighs the intrinsic substitutability between the public and private signals in revealing information to the investors.

# 5 An Application to Private Word-of-Mouth Communications

Our model can be used to study investors' private word-of-mouth communications that have been increasingly relevant in light of technological innovations. The New York Stock Exchange (NYSE) recently noted that "social media has become a crucial source of information for the financial services community."[11] A natural question is how firms' public disclosure would react in response to the change of information environment.

We use the technology developed by Duffie and Manso (2007) and Duffie et al. (2009) to model investors' private word-of-mouth communications. In particular, each investor meets other investors (e.g., family members or friends) at a sequence of Poisson arrival time with a mean arrival rate $\lambda \geq 0$ that is exogenous and common across all investors. When two investors meet, they exchange their posterior beliefs about the firm value $v$. Given the joint-normal information structure, it is sufficient for the purpose of updating investors' beliefs about $v$ that each investor $i$ tells her counterpart

[11] "NYSE Technologies and SMA to Distribute Social Media Analysis Data via SFTI" on NYSE Technologies (https://nysetechnologies.nyx.com).

$j$, at each meeting, her current conditional mean $\tilde{\mu}_i$ and the total number of signals $N_i$ from which $\tilde{\mu}_i$ is derived. The number $N_i$ is initially one at $t = 1$ (i.e., the endowed private signal), and then increases at each meeting by the number of signals $N_j$ gathered by her counterpart $j$ prior to the meeting. The word-of-mouth communications take place continuously prior to the trading date $t = 2$. According to Andrei and Cujean (2017, Proposition 1), the cross-sections distribution of the number of signals $N_i$ at the trading date is:[12]

$$\pi(n) = \begin{cases} e^{-\lambda} & \text{if } n = 1, \\ e^{-(n-1)\lambda}(e^{\lambda} - 1)^{n-2}(1 - e^{-\lambda}) & \text{if } n = 2, 3, 4, \dots . \end{cases} \tag{29}$$

A higher poisson arrival rate $\lambda$ corresponds to more active word-of-mouth communications ($\lambda = 0$ means that no one shares information, as in the main model). As is clear from (29), modeling word-of-mouth communications inevitably result in asymmetrically informed investors in terms of heterogenous signal precision. This added information asymmetry complicates the analysis in our main model. Nonetheless, Lemma 2 shows that our previous results are qualitatively unaffected.

**Lemma 2 (Equilibrium with Word of Mouth)** *All results derived in the main model are preserved once we replace the private signal precision $\tau_\eta$ with the cross-sectional average precision $\bar{\mathbf{N}}\tau_\eta$ after word of mouth. The constant $\bar{\mathbf{N}}$ is the cross-sectional average number of signals that investors accumulate by the time of trading:*

$$\bar{\mathbf{N}} = \sum_{n=1,2,3\dots} n\,\pi(n) = e^{\lambda}. \tag{30}$$

[12]Andrei and Cujean (2017) derive the distribution of the number of *incremental* signals, while (29) is the distribution of the total number of signals, including each investor's signal endowment $y_i$.

Lemma 2 shows that more active word-of-mouth communications (i.e., higher $\lambda$) affect our previous equilibrium analysis by increasing the cross-sectional average precision of investors' private signals. This average-precision only feature has been shown in the literature (e.g., Lambert et al., 2011) to be a standard feature of the classical noisy rational expectation models. Given the monotonic relation between word-of-mouth communications $\lambda$ and the cross-sectional average precision $\bar{\mathbf{N}}\tau_\eta$, we can apply Proposition 4 to obtain the following result.[13]

**Corollary 1** *Active word-of-mouth communications will lead to more public disclosure if and only if the variance of the noisy supply is large. That is, for any $\tau_x^* > 0$,*

$$\frac{d}{d\lambda}\tau_x^* > 0 \text{ if and only if } \sigma_\varepsilon^2 > 2\bar{\mathbf{N}}\tau_\eta r^2. \tag{31}$$

Corollary 1 sheds light on the debate over the implications of word-of-mouth communications (or other private information channels) on firms' disclosures. Some claim that investors' private communications will crowd out firms' public disclosures. By facilitating private information discovery, word of mouth improves investors' private information, and thereby lowers their reliance on firms' public disclosures according to Bayes' rule. Opponents point out that, instead of making investors better informed, word-of-mouth communications introduce misleading rumors that can cause mispricing if investors have bounded rationality. They argue that firms should disclosure more to mitigate the damage caused by rumors disseminated via word of mouth. Our analyses cast doubt on the reasoning of both sides. On the one hand, we show that even if word of mouth improves investors' private information and lowers their reliance

[13] Upon discussing the word-of-mouth application, we confine attention to the more interesting case in which the disclosure precision is $\tau_x^* > 0$. Proposition 3 provides the necessary and sufficient condition.

on the firm's disclosures, that does not necessarily translate into a lower provision of public disclosures. On the other hand, while we show that word of mouth can indeed drive the price (ex post) away from its fundamental value and, hence, induce more public disclosure, our results arise *without* assuming any biased rumors or bounded rationality. That is, we show a "dark side" of word of mouth arises even under the benevolent assumption that such private communications are unbiased and truthful.[14]

**Empirical Implications** The model also yields two sets of empirical predictions. First, we can conduct comparative statics to the critical threshold $\Sigma \doteq 2\bar{\mathbf{N}}\tau_\eta r^2$ above which investors' private word of mouth leads to more precise public disclosures. The result is helpful in predicting cross-sectionally which types of firms are more likely to improve (or lower) their public disclosure quality when technological innovations such as social media facilitate private information sharing. The idea is that while econometricians do not observe the exact value of $\sigma_\varepsilon^2$, they know that $\sigma_\varepsilon^2 > \Sigma$ is more likely to satisfy if $\Sigma$ becomes smaller.

**Proposition 5 (Disclosure Response)** *A firm is more likely to improve its disclosure quality in response to investors' more active private word-of-mouth communications if:*

*(i) Investors are more risk averse:* $\frac{d}{d\,r^{-1}}\Sigma < 0$, *or*

*(ii) Investors' initial private signal endowment is noisier:* $\frac{d}{d\sigma_\eta}\Sigma < 0$.

Our second empirical prediction speaks to the effect of word-of-mouth communications on market depth, after endogenizing the precision of public disclosure

[14] Bagnoli and Watts (2017) study a voluntary disclosure model in a risk-neutral setup. They show that negative pressures (exogenous event that results in the market reducing its expectation of the firm value) can force the firm to disclose information that the firm withheld initially.

$\tau_x^*$. We use the inverse of coefficient $\alpha_\varepsilon$ in the pricing function (8) to measure the market depth (as in, for example, Vives, 2010; Han and Yang, 2013). The idea behind the measure, as argued in Vives (2010), is that a change in noise trading by one unit moves prices by $\alpha_\varepsilon$; a market is deep if a noise trader shock is absorbed without moving prices much, which happens when $\alpha_\varepsilon$ is low.

**Proposition 6 (Lowering Market Depth)** *More active word-of-mouth communications by investors reduce the market depth; that is,* $\frac{d}{d\lambda}\alpha_\varepsilon^{-1} < 0$ *for* $\forall \tau_x^* > 0$.

This result is in contrast to the non-monotonic relation shown in the literature. For example, Grossman and Stiglitz (1980) state in their Conjecture 7 that market depth $\alpha_\epsilon^{-1}$ is non-monotonic in the quality of private information. It is also worth comparing Proposition 6 to a well-known result that more precise private information can lower market liquidity if the private information exacerbates information asymmetry among investors (e.g., Verrecchia, 2001).[15] The mechasism behind our liquidity-reducing result in Proposition 6 is entirely different because it does not require information asymmetry. In fact, we can completely eliminate information asymmetry by letting *all* investors' private signal precision be $\tau_\eta$ and derive the monotonic decrease in market depth, i.e., $\frac{d}{d\tau_\eta}\alpha_\varepsilon^{-1} < 0$. The reason we obtain an unambiguous negative correlation is that our model endogenizes the firm's disclosure quality $\tau_x^*$. Allowing the manager to adjust the firm's disclosure quality is descriptive. When testing Proposition 6, however, cautions should be given if a firm is in a specific industry or a sensitive period in which its manager has little or no discretion over disclosure quality. In this case, the disclosure quality $\tau_x$ should be treated as given exogenously, and more active word of mouth $\lambda$

[15] Kim and Verrecchia (1994) show how disclosure can reduce liquidity by affecting information asymmetry. Caskey et al. (2015) study the effect on bid-ask spread of information dissemination in networks in a sequential trade model á la Glosten and Milgrom (1985).

would have a non-monotonic effect on market depth as seen in the literature.

# 6 Conclusion

Technological innovations such as social media greatly facilitate individual investors' private information discoveries and communications. In this paper, we examine firms' responses to such changes. Contrary to the casual intuition that more precise private information compete with and crowd out firms' public disclosures, we show that firms often commit to more precise public disclosures when the investors' private information becomes more precise. This complementarity arises because, as investors' private information becomes more precise, the information aggregation role of price becomes so efficient that investors choose to rely more heavily on pricing in valuing the firm. This increased reliance on price has a side effect of amplifying the price impact of noisy supplies, which in turn strengthens the value of public disclosures because more precise disclosure helps mitigate such price impact and, hence, the noise-driven volatilities. We show that when the asset supply is expected to be volatile, the call for more precise public disclosures to lower the otherwise exacerbated noise-driven volatility outweighs the intrinsic substitutability between the public and private signals in revealing information to the investors.

We apply our model to study investors' private word-of-mouth communications. The analysis demonstrates that there is an endogenous "dark side" to such private communications that acts to amplify noisy supplies and price volatilities even when the communications are assumed to be unbiased and truthful. Our results suggest that a firm is more likely to increase the quality of its public disclosure in response to more active private word of mouth by investors if investors are more risk averse or if their private information endowments are imprecise. We also show that more active word of mouth lowers the market depth after endogenizing the firm's public disclosure. These empirical predictions are relevant in light of the recent debate on the consequence of technological innovations that facilitate investors' interpersonal communications.

The manager's risk aversion plays an important role in our analysis (in particular, his disutility when facing a more volatile price). In a way, our emphasis on the manager's utility is in line with Beyer et al. (2010, p. 305) who state in their review of the disclosure literature that "[i]t is management and not the 'firm' that makes disclosure decisions. As a result, the costs and benefits of disclosure that explain disclosure decisions reflect management's utility and disutility from making a disclosure." According to Beyer et al. (2010), "[m]ost models assume that the managers attempt to maximize share price." Our model complements previous studies by also considering the manager's disutility that is associated with price volatility. While we acknowledge that some managers may prefer a more volatile price, the risk-aversion assumption and, in our opinion, the incentives to avoid price volatilities are descriptive in many cases. For future research, it seems to be an interesting avenue to study the interactions between firms' disclosures and the investors' private communications in a dynamic setting.

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# A Appendix: Proofs

**Proof of Proposition 1:** We organize the proof in two steps. In the first step, we reason from the investors' perspective: we take the investors' conjecture $\hat{a}$ about the manager's effort as given, and solve for the linear pricing function that clears the market (given the conjecture $\hat{a}$). We then reason from the manager's perspective in the second step: we take the linear pricing function derived in the first step as given and solve for the manager's optimal effort choice $a^*$. The rationality condition ensures that $\hat{a} = a^*$ in equilibrium.

**Step 1:** We guess and verify the following linear pricing equilibrium:

$$p = \hat{\alpha}_0 + \hat{\alpha}_v v + \hat{\alpha}_x \zeta - \hat{\alpha}_\varepsilon \varepsilon, \tag{A.1}$$

where the coefficients can depend on the investors' conjecture $\hat{a}$ (among other primitives of the model) but not on the manager's actual effort $a$, which is unobservable by its nature.

Consider the demand of the risky asset from any investor $i$ who observes (i) the public signal $x$, (ii) the market price $p$, and (iii) an independent private signal $y_i = v + \eta_i$ prior to trading. The market price $p$ is informationally equivalent to

$$q \doteq \frac{p - \hat{\alpha}_x x - \hat{\alpha}_0}{\hat{\alpha}_v - \hat{\alpha}_x} = \frac{p - \hat{\alpha}_x (v + \zeta) - \hat{\alpha}_0}{\hat{\alpha}_v - \hat{\alpha}_x} = v - \frac{\hat{\alpha}_\varepsilon}{\hat{\alpha}_v - \hat{\alpha}_x} \varepsilon, \tag{A.2}$$

which is a noisy signal of firm value $v$ with precision $\left( \frac{\hat{\alpha}_v - \hat{\alpha}_x}{\hat{\alpha}_\varepsilon} \right)^2 \tau_\varepsilon$. Note that $q$ is easier to work with because its mean is $v$. We can express investor $i$'s information set as $\mathcal{F}_i = \{y_i, x, q, \hat{a}\}$, where $\hat{a}$ is the investors' conjecture of the manager's effort. The joint normality implies that

$$\operatorname{var}\left(v|\mathcal{F}_i\right) = \frac{1}{\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta}, \tag{A.3}$$

$$\mathbb{E}\left(v|\mathcal{F}_i\right) = \frac{\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon q + \tau_x x + \tau_\phi \hat{a} + \tau_\eta y_i}{\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta}. \tag{A.4}$$

Therefore, investor $i$'s demand for the risky-asset is

$$\begin{aligned} D_i &= \frac{r\Big(\mathbb{E}\left(v|\mathcal{F}_i\right) - p\Big)}{\operatorname{var}\left(v|\mathcal{F}_i\right)} \\ &= r\Bigg[\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon\frac{p-\hat{\alpha}_x x-\hat{\alpha}_0}{\hat{\alpha}_v-\hat{\alpha}_x} + \tau_x x + \tau_\phi \hat{a} + \tau_\eta y_i \\ &\quad -p\left(\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta\right)\Bigg]. \end{aligned} \tag{A.5}$$

Integrating $D_i$ over the continuum of investors and making use of the market-clearing condition $\int_i D_i di = \varepsilon$, we can show the following:

$$\begin{aligned} & r\Bigg[\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon\frac{p-\hat{\alpha}_x x-\hat{\alpha}_0}{\hat{\alpha}_v-\hat{\alpha}_x} + \tau_x x + \tau_\phi \hat{a} + \int_i \tau_\eta y_i di \\ &\quad -p\left(\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta\right)\Bigg] = \varepsilon, \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \quad & r\Bigg[\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon\frac{p-\hat{\alpha}_x\left(v+\zeta\right)-\hat{\alpha}_0}{\hat{\alpha}_v-\hat{\alpha}_x} + \tau_x\left(v+\zeta\right) + \tau_\phi \hat{a} + \tau_\eta v \\ &\quad -p\left(\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2\tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta\right)\Bigg] = \varepsilon, \end{aligned}$$

from which we know the market-clearing price is

$$p = \frac{-\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{\hat{\alpha}_x\zeta+\hat{\alpha}_0}{\hat{\alpha}_v-\hat{\alpha}_x} + \tau_x\zeta + \tau_\phi\hat{a} + \left(\tau_x + \tau_\eta - \left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{\hat{\alpha}_x}{\hat{\alpha}_v-\hat{\alpha}_x}\right) v - \frac{\varepsilon}{r}}{\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta - \left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{1}{\hat{\alpha}_v-\hat{\alpha}_x}}. \quad \text{(A.6)}$$

To determine the coefficients, we impose the rational condition that the conjectured pricing function (A.1) coincides with the true market-clearing price (A.6) in equilibrium. That is, the coefficients satisfy:

$$\hat{\alpha}_0 = \frac{-\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{\hat{\alpha}_0}{\hat{\alpha}_v-\hat{\alpha}_x} + \tau_\phi\hat{a}}{\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta - \left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{1}{\hat{\alpha}_v-\hat{\alpha}_x}}, \quad \text{(A.7)}$$

$$\hat{\alpha}_v = \frac{\tau_x + \tau_\eta - \left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{\hat{\alpha}_x}{\hat{\alpha}_v-\hat{\alpha}_x}}{\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta - \left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{1}{\hat{\alpha}_v-\hat{\alpha}_x}}, \quad \text{(A.8)}$$

$$\hat{\alpha}_x = \frac{-\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{\hat{\alpha}_x}{\hat{\alpha}_v-\hat{\alpha}_x} + \tau_x}{\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta - \left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{1}{\hat{\alpha}_v-\hat{\alpha}_x}}, \quad \text{(A.9)}$$

$$\hat{\alpha}_\varepsilon = \frac{\frac{1}{r}}{\left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon + \tau_x + \tau_\phi + \tau_\eta - \left(\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon}\right)^2 \tau_\varepsilon \frac{1}{\hat{\alpha}_v-\hat{\alpha}_x}}. \quad \text{(A.10)}$$

It is easy to verify that $\frac{\hat{\alpha}_v-\hat{\alpha}_x}{\hat{\alpha}_\varepsilon} = \tau_\eta r$. We can then simplify (A.2) as $q = v - \frac{\varepsilon}{r\tau_\eta}$ and verify that $\tau_p = \left(\tau_\eta r\right)^2 \tau_\varepsilon$ is the precision of market price $p$, which is informationally equivalent to $q$. The system of linear equations shown above determines the pricing coefficients:

$$\hat{\alpha}_0 = \frac{\tau_\phi}{\tau_\phi + \tau_x + \tau_\eta + \tau_p} \times \hat{a}, \tag{A.11}$$

$$\hat{\alpha}_v = \frac{\tau_x + \tau_\eta + \tau_p}{\tau_\phi + \tau_x + \tau_\eta + \tau_p}, \tag{A.12}$$

$$\hat{\alpha}_x = \frac{\tau_x}{\tau_\phi + \tau_x + \tau_\eta + \tau_p}, \tag{A.13}$$

$$\hat{\alpha}_\varepsilon = \frac{1}{r\tau_\eta} \frac{\tau_\eta + \tau_p}{\tau_\phi + \tau_x + \tau_\eta + \tau_p}, \tag{A.14}$$

where $\hat{a}$ is the investor's conjecture about the manager's unobservable effort $a$.

**Step 2:** We next solve for the manager's equilibrium effort choice. In particular, the manager takes the linear pricing function charactered above as given and chooses $a$ to maximize his certainty equivalent:

$$\max_a \mathbb{E}[p|a, \hat{a}] - C(a) - \frac{\rho}{2}\text{var}(p|a, \hat{a}), \tag{A.15}$$

where $\mathbb{E}[p|a, \hat{a}] = \hat{\alpha}_0 + \hat{\alpha}_v \times a$ and $\text{var}(p|a, \hat{a}) = (\hat{\alpha}_v)^2 \sigma_\phi^2 + (\hat{\alpha}_x)^2 \sigma_x^2 + (\hat{\alpha}_\varepsilon)^2 \sigma_\varepsilon^2$ follow from the linear pricing function characterized in Step 1. Inspecting the first-order condition yields the manager's best response as follows:

$$a^*(\tau_x) = \hat{\alpha}_v = \frac{\tau_x + \tau_\eta + \tau_p}{\tau_\phi + \tau_x + \tau_\eta + \tau_p} = 1 - \tau_\phi \text{var}(v|\mathcal{F}), \tag{A.16}$$

where $\text{var}(v|\mathcal{F}) = \frac{1}{\tau_\phi + \tau_x + \tau_\eta + \tau_p}$ is investor $i$'s residual uncertainty about $v$ given her information set $\mathcal{F} = \{y_i, x, p, \hat{a}\}$. Since the manager's best response is independent of the investors' conjecture $\hat{a}$, $\hat{a}$ must equal $a^*(\tau_x) = 1 - \tau_\phi \text{var}(v|\mathcal{F})$ to be correct in equilibrium. Finally, replacing the conjectured effort $\hat{a}$ in (A.11) with the equilibrium effort $a^*$ yields the equilibrium linear pricing coefficients $(\alpha_0, \alpha_v, \alpha_x, \alpha_\varepsilon)$ shown in the proposition. ■

**Proof of Lemma 1:** Denote by $\mathcal{F}_i = \{y_i, x, p, \hat{a}\}$ investor $i$'s information set, where $p$ is informationally equivalent to $q = \frac{p-\alpha_x x-\alpha_0}{a_v-a_x} = v - \frac{\varepsilon}{r\tau_\eta}$, It follows from Bayes rule that

$$\mathbb{E}\left(v|\mathcal{F}_i\right) = \frac{\tau_p q + \tau_x x + \tau_\phi \hat{a} + \tau_\eta y_i}{\tau_\phi + \tau_x + \tau_\eta + \tau_p}, \tag{A.17}$$

which can be denoted as $\mathbb{E}\left(v|\mathcal{F}_i\right) = w_0\hat{a} + w_q q + w_x x + w_y y_i$, where $w_q = \frac{\tau_p}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}$, $w_0 = \frac{\tau_\phi}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}$, $w_x = \frac{\tau_x}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}$, and $w_y = \frac{\tau_\eta}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}$.

It is easy to verify

$$\begin{aligned}\frac{dw_q}{d\tau_\eta} &= \frac{\frac{d\tau_p}{d\tau_\eta}\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right) - \tau_p\left(1 + \frac{d\tau_p}{d\tau_\eta}\right)}{\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right)^2} \\ &= \frac{2\frac{\tau_p}{\tau_\eta}\left(\tau_\phi + \tau_x\right) + \tau_p}{\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right)^2} > 0,\end{aligned} \tag{A.18}$$

where we use $\frac{d\tau_p}{d\tau_\eta} = \frac{d\left(\tau_\eta r\right)^2 \tau_\varepsilon}{d\tau_\eta} = 2r^2\tau_\eta\tau_\varepsilon = 2\frac{\tau_p}{\tau_\eta}$. We also obtain

$$\frac{dw_q}{d\tau_x} = -\frac{\tau_p}{\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right)^2} < 0. \tag{A.19}$$

**Proof of Proposition 2:** Using the market clearing condition $\int_i D_i di = \varepsilon$ and the investor $i$'s demand function $D_i = \frac{r\left(\mathbb{E}\left(v|\mathcal{F}_i\right)-p\right)}{\mathrm{var}\left(v|\mathcal{F}_i\right)}$, we can rewrite the pricing function as

$$p = \int_i \mathbb{E}\left(v|\mathcal{F}_i\right) di - \frac{\mathrm{var}\left(v|\mathcal{F}\right)}{r}\varepsilon, \tag{A.20}$$

where $\text{var}\left(v|\mathcal{F}\right) = \frac{1}{\tau_\phi+\tau_x+\tau_\eta+\tau_p}$ is investors' conditional variance. Substituting $\mathbb{E}\left(v|\mathcal{F}_i\right)$ from (A.17) into (A.20) and using the fact $\int_i y_i d_i = v$, we simplify (A.20) to $p = \left[w_0\hat{a} + w_x x + w_y v + w_q(v - \frac{\epsilon}{r\tau_\eta})\right] - \frac{\text{var}\left(v|\mathcal{F}\right)}{r}\varepsilon$, which allows us to show

$$\begin{aligned}
\alpha_\epsilon = \left|\frac{dp}{d\varepsilon}\right| =& \frac{\text{var}\left(v|\mathcal{F}\right)}{r} + \left|\frac{d}{d\varepsilon}\int_i \mathbb{E}\left(v|\mathcal{F}_i\right) di\right| \\
=& \frac{\text{var}\left(v|\mathcal{F}\right)}{r} + \frac{w_q}{r\tau_\eta}.
\end{aligned} \tag{A.21}$$

It is straightforward to verify that

$$\begin{aligned}
\frac{d}{d\tau_x}\alpha_\varepsilon &= \frac{1}{r}\frac{d}{d\tau_x}\text{var}\left(v|\mathcal{F}\right) + \frac{1}{r\tau_\eta}\frac{d}{d\tau_x}w_q \\
&= -\frac{1}{r}\text{var}^2\left(v|\mathcal{F}\right) - \frac{1}{r\tau_\eta}\frac{\tau_p}{\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right)^2} \\
&= -\frac{1 + \frac{\tau_p}{\tau_\eta}}{r\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right)^2} < 0.
\end{aligned} \tag{A.22}$$

Similarly, we show

$$\begin{aligned}
\frac{d}{d\tau_\eta}\alpha_\varepsilon &= \frac{1}{r}\frac{d}{d\tau_\eta}\text{var}\left(v|\mathcal{F}\right) + \frac{1}{r}\frac{d}{d\tau_\eta}\left(\frac{w_q}{\tau_\eta}\right) \\
&= -\frac{1}{r}\text{var}^2\left(v|\mathcal{F}\right)\left(1 + 2\frac{\tau_p}{\tau_\eta}\right) + \frac{\frac{\tau_p}{\tau_\eta}\frac{\tau_\phi+\tau_x-\tau_p}{\tau_\eta}}{r\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right)^2} \\
&= \frac{-1 + \frac{\tau_p}{\tau_\eta}\frac{\tau_\phi+\tau_x-2\tau_\eta-\tau_p}{\tau_\eta}}{r\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right)^2} \\
&= -\frac{\left(r^2\tau_\eta\tau_\varepsilon\right)^2 - r^2\tau_\varepsilon\left(\tau_\phi + \tau_x - 2\tau_\eta\right) + 1}{r\left(\tau_\phi + \tau_x + \tau_\eta + \tau_p\right)^2},
\end{aligned} \tag{A.23}$$

which is positive if and only if $\tau_\varepsilon \in (\underline{\tau}_\varepsilon, \bar{\tau}_\varepsilon)$, where

$$\underline{\tau}_\varepsilon \doteq \frac{\left(\tau_\phi + \tau_x - 2\tau_\eta\right) - \sqrt{\left(\tau_\phi + \tau_x\right)\left(\tau_\phi + \tau_x - 4\tau_\eta\right)}}{r^2\tau_\eta^2}, \tag{A.24}$$

$$\bar{\tau}_\varepsilon \doteq \frac{\left(\tau_\phi + \tau_x - 2\tau_\eta\right) + \sqrt{\left(\tau_\phi + \tau_x\right)\left(\tau_\phi + \tau_x - 4\tau_\eta\right)}}{r^2\tau_\eta^2}. \tag{A.25}$$

The sets $(\underline{\tau}_\varepsilon, \bar{\tau}_\varepsilon)$ is not empty if and only if $\tau_\phi + \tau_x > 4\tau_\eta$. Collecting the conditions proves the result. ■

**Proof of Proposition 3 :** Substituting $a^*(\tau_x) = 1 - \tau_\phi \text{var}\left(v|\mathcal{F}\right)$ from Proposition 1, we can express the manager's objective function $U^M$ as:

$$\begin{aligned} U^M &= \mathbb{E}[p|a^*] - C(a^*) - \frac{\rho}{2}\text{var}(p) \\ &= a^* - \frac{(a^*)^2}{2} - \frac{\rho}{2}\left(V_F + V_N\right) \\ &= \frac{1}{2} - \frac{\tau_\phi^2}{2}\text{var}^2\left(v|\mathcal{F}\right) - \frac{\rho}{2}\left(\text{var}\left[\mathbb{E}(p|v)\right] + \mathbb{E}\left[\text{var}\left(p|v\right)\right]\right), \end{aligned} \tag{A.26}$$

where the last equality uses the result $a^* = \alpha_v = 1 - \tau_\phi \text{var}\left(v|\mathcal{F}\right)$ and the mathematical observation of $\text{var}(p) = \text{var}\left[\mathbb{E}(p|v)\right] + \mathbb{E}\left[\text{var}\left(p|v\right)\right]$. Using the pricing function, we can rewrite the *Fundamental-driven volatility* $V_F$ as

$$V_F \doteq \text{var}\left[\mathbb{E}(p|v)\right] = \alpha_v^2/\tau_\phi = \left(1 - \tau_\phi \text{var}(v|\mathcal{F})\right)^2 \frac{1}{\tau_\phi}, \tag{A.27}$$

and the *Noise-driven volatility* $V_N$ as

$$V_N \doteq \mathbb{E}\left[\mathrm{var}\left(p|v\right)\right] = \alpha_x^2/\tau_x + \alpha_\varepsilon^2/\tau_\varepsilon = \mathrm{var}^2(v|\mathcal{F})\left(\tau_x + \frac{\left(1+\frac{\tau_p}{\tau_\eta}\right)^2}{r^2\tau_\varepsilon}\right). \tag{A.28}$$

Differentiating the manager's certainty equivalent (A.26) with respect to $\tau_x$, we obtain

$$\begin{aligned}\frac{dU^M}{d\tau_x} &= \underbrace{\tau_\phi^2\mathrm{var}^3(v|\mathcal{F})}_{MB \text{ on Effort}} + \underbrace{\overbrace{\rho\left(\tau_x\mathrm{var}\left(v|\mathcal{F}\right) - \frac{1}{2}\right)\mathrm{var}^2(v|\mathcal{F})}^{-\frac{\rho}{2}\frac{d(\alpha_x^2/\tau_x)}{d\tau_x}} + \overbrace{\rho\frac{\left(1+\frac{\tau_p}{\tau_\eta}\right)^2}{r^2\tau_\varepsilon}\mathrm{var}^3(v|\mathcal{F})}^{-\frac{\rho}{2}\frac{d(\alpha_\varepsilon^2/\tau_\varepsilon)}{d\tau_x}}}_{\text{Marginal Effect on } V_N} \\ &\quad - \underbrace{\rho\left(1-\tau_\phi\mathrm{var}\left(v|\mathcal{F}\right)\right)\mathrm{var}^2(v|\mathcal{F})}_{MC \text{ on } V_F},\end{aligned} \tag{A.29}$$

where we use $\frac{\partial\mathrm{var}\left(v|\mathcal{F}\right)}{\partial\tau_x} = -\mathrm{var}^2\left(v|\mathcal{F}\right)$ in the derivation. One can rewrite the condition above to obtain the following first-order-condition:

$$FOC \equiv \mathrm{var}^2(v|\mathcal{F})\left\{\left[\tau_\phi^2 + \rho\tau_\phi + \rho\left(\tau_x + \frac{\left(1+\frac{\tau_p}{\tau_\eta}\right)^2}{r^2\tau_\varepsilon}\right)\right]\mathrm{var}(v|\mathcal{F}) - \frac{3\rho}{2}\right\} = 0. \tag{A.30}$$

Plugging $\mathrm{var}\left(v|\mathcal{F}\right) = \frac{1}{\tau_x+\tau_\eta+\tau_p+\tau_\phi}$ and $\tau_p = \left(\tau_\eta r\right)^2\tau_\varepsilon$ into $FOC$, we solve for the optimal precision:

$$\tau_x^* = 2\frac{1}{r^2\tau_\varepsilon} + \frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \tau_\eta - \left(\tau_\eta r\right)^2\tau_\varepsilon, \tag{A.31}$$

with $\tau_x^* > 0$ if and only if $\sigma_\varepsilon^2 = \frac{1}{\tau_\varepsilon} > \frac{r^2\left[\sqrt{\left(\frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \tau_\eta\right)^2 + 8\tau_\eta^2} - \left(\frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \tau_\eta\right)\right]}{4} \equiv \underline{\Sigma}_I\left(\tau_\eta\right)$. To complete the proof, we verify the second-order condition of the maximization problem is negative. That is, $\frac{d^2U^M}{d\tau_x^2}|_{\tau_x=\tau_x^*} = \frac{dFOC}{d\tau_x} = -\frac{\rho}{2}\text{var}^3(v|\mathcal{F}) < 0$. ■

**Proof of Proposition 4:** We present our proof in two parts. We first apply the implicit function theorem to analyze $\frac{\partial\tau_x^*}{\partial\tau_\eta}$, taking as given that the precision choice is interior, i.e., $\tau_x^* > 0$. In Part II, we remove the restriction of $\tau_x^* > 0$ and take into account that $\tau_\eta$ affects whether $\tau_x^*$ is strictly positive, as shown in Proposition 3.

**Part I** (Taking $\tau_x^* > 0$ as given): Denote by $FOC$ the first-order condition at the optimal $\tau_x^*$, i.e., equation (A.30). We apply the implicit function theorem to obtain

$$\frac{d\tau_x^*}{d\tau_\eta} = -\frac{\frac{dFOC}{d\tau_\eta}}{\frac{dFOC}{d\tau_x}} = -\frac{\frac{\partial FOC}{\partial\text{var}\left(v|\mathcal{F}\right)}\frac{\partial\text{var}\left(v|\mathcal{F}\right)}{\partial\tau_\eta} + \frac{\partial FOC}{\partial\frac{\tau_p}{\tau_\eta}}\frac{\partial\frac{\tau_p}{\tau_\eta}}{\partial\tau_\eta}}{\frac{dFOC}{d\tau_x}}, \tag{A.32}$$

where we have shown $\frac{dFOC}{d\tau_x}|_{\tau_x=\tau_x^*} = -\frac{\rho}{2}\text{var}^3(v|\mathcal{F}) < 0$.

First, the optimality of $\tau_x^*$ allows us to further simplify $\frac{\partial FOC}{\partial\text{var}\left(v|\mathcal{F}\right)}$ as:

$$\begin{aligned}
&\frac{\partial FOC}{\partial\text{var}\left(v|\mathcal{F}\right)}|_{\tau_x=\tau_x^*}\\
&= 3\left[\tau_\phi^2 + \rho\tau_\phi + \rho\left(\tau_x + \frac{\left(1+\frac{\tau_p}{\tau_\eta}\right)^2}{r^2\tau_\varepsilon}\right)\right]\text{var}^2(v|\mathcal{F}) - 3\rho\text{var}(v|\mathcal{F})\\
&= 3\left\{\left[\tau_\phi^2 + \rho\tau_\phi + \rho\left(\tau_x + \frac{\left(1+\frac{\tau_p}{\tau_\eta}\right)^2}{r^2\tau_\varepsilon}\right)\right]\text{var}^2(v|\mathcal{F}) - \frac{3\rho}{2}\text{var}(v|\mathcal{F})\right\} + \frac{3\rho}{2}\text{var}(v|\mathcal{F})\\
&= \frac{3\rho}{2}\text{var}(v|\mathcal{F}) > 0,
\end{aligned} \tag{A.33}$$

where the last equality follows from the first-order condition (A.30). Combining the equality above with $\frac{d\mathrm{var}(v|\mathcal{F})}{d\tau_\eta} = -\mathrm{var}^2(v|\mathcal{F})\left(1+2\frac{\tau_p}{\tau_\eta}\right)$, we obtain

$$\frac{\partial FOC}{\partial \mathrm{var}(v|\mathcal{F})}\frac{\partial \mathrm{var}(v|\mathcal{F})}{\partial \tau_\eta}|_{\tau_x=\tau_x^*} = -\frac{3\rho}{2}\mathrm{var}^3(v|\mathcal{F})\left(1+2\frac{\tau_p}{\tau_\eta}\right). \tag{A.34}$$

Second, note that the only relevant part of $FOC$ (in differentiating $\frac{\tau_p}{\tau_\eta}$) comes from the marginal effect of $\tau_x$ on $\alpha_\varepsilon^2\sigma_\varepsilon^2$ as seen in (A.29). Therefore, we obtain

$$\begin{aligned}\frac{\partial FOC}{\partial \frac{\tau_p}{\tau_\eta}}\frac{\partial \frac{\tau_p}{\tau_\eta}}{\partial \tau_\eta} = \frac{\partial\left(\frac{d(-\frac{\rho}{2}\alpha_\varepsilon^2\sigma_\varepsilon^2)}{d\tau_x}\right)}{\partial \frac{\tau_p}{\tau_\eta}}\frac{\partial \frac{\tau_p}{\tau_\eta}}{\partial \tau_\eta} &= \frac{\partial\left(\rho\frac{\left(1+\frac{\tau_p}{\tau_\eta}\right)^2}{r^2\tau_\varepsilon}\mathrm{var}^3(v|\mathcal{F})\right)}{\partial \frac{\tau_p}{\tau_\eta}}\frac{\partial \frac{\tau_p}{\tau_\eta}}{\partial \tau_\eta} \\ &= 2\rho\left(1+\frac{\tau_p}{\tau_\eta}\right)\mathrm{var}^3(v|\mathcal{F}).\end{aligned} \tag{A.35}$$

Overall, we can show that

$$\begin{aligned}\frac{d\tau_x^*}{d\tau_\eta} &= -\frac{\frac{dFOC}{d\tau_\eta}|_{\tau_x=\tau_x^*}}{\frac{dFOC}{d\tau_x}|_{\tau_x=\tau_x^*}} = -\frac{\frac{\partial FOC}{\partial \mathrm{var}\left(v|\mathcal{F}\right)}\frac{\partial \mathrm{var}\left(v|\mathcal{F}\right)}{\partial \tau_\eta}|_{\tau_x=\tau_x^*} + \frac{\partial FOC}{\partial \frac{\tau_p}{\tau_\eta}}\frac{\partial \frac{\tau_p}{\tau_\eta}}{\partial \tau_\eta}}{\frac{dFOC}{d\tau_x}|_{\tau_x=\tau_x^*}} \\ &= -\frac{-\frac{3\rho}{2}\mathrm{var}^3(v|\mathcal{F})\left(1+2\frac{\tau_p}{\tau_\eta}\right)+2\rho\left(1+\frac{\tau_p}{\tau_\eta}\right)\mathrm{var}^3(v|\mathcal{F})}{-\frac{\rho}{2}\mathrm{var}^3(v|\mathcal{F})} \\ &= \underbrace{-3(1+2\frac{\tau_p}{\tau_\eta})}_{\text{Information Revelation Effect (-)}} + \underbrace{4(1+\frac{\tau_p}{\tau_\eta})}_{\text{Inference Effect (+)}} \\ &= 1-2\frac{\tau_p}{\tau_\eta}.\end{aligned} \tag{A.36}$$

It follows immediately from (A.36) that $\frac{d\tau_x^*}{d\tau_\eta} > 0$ if and only if $\frac{\tau_p}{\tau_\eta} < \frac{1}{2}$, which is equivalent to $\sigma_\varepsilon^2 > 2\tau_\eta r^2$ as claimed in the proposition (recall $\frac{\tau_p}{\tau_\eta} = r^2\tau_\eta\tau_\varepsilon$).

**Part II** (Allowing for any $\tau_x^* \geq 0$ ): The proof in Part I assumes the solution is interior, i.e., $\tau^* > 0$. This implicit assumption is satisfied if the condition $\sigma_\varepsilon^2 > 2\tau_\eta r^2$ shown above ensures/implies the condition $\sigma_\varepsilon^2 > \underline{\Sigma}_I$ for an interior solution characterized in Proposition 3. One can show that $\sigma_\varepsilon^2 > 2\tau_\eta r^2$ ensures an interior $\tau_x^* > 0$ if and only if the following holds:

$$\begin{aligned}\underline{\Sigma}_C\left(\tau_\eta\right) &\equiv 2\tau_\eta r^2 \\ &\geq \frac{r^2\left[\sqrt{\left(\frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \tau_\eta\right)^2 + 8\tau_\eta^2} - \left(\frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \tau_\eta\right)\right]}{4} \equiv \underline{\Sigma}_I\left(\tau_\eta\right), \end{aligned} \tag{A.37}$$

where $\underline{\Sigma}_C$ and $\underline{\Sigma}_I$ are the lower bound of $\sigma_\varepsilon^2$ required by the complementarity result and interior solution, respectively. The inequality (A.37) is equivalent to $\tau_\eta \geq \frac{2}{9}\left(\tau_\phi - \frac{2}{\rho}\tau_\phi^2\right)$, which holds easily if $\tau_\phi - \frac{2}{\rho}\tau_\phi^2 \leq 0$, or equivalently, $\rho \leq 2\tau_\phi$. That is, whenever $\rho \leq 2\tau_\phi$, $\sigma_\varepsilon^2 > 2\tau_\eta r^2$ ensures $\tau_x^* > 0$ and, hence, is a necessary and sufficient condition for $\frac{d}{d\tau_\eta}\tau_x^* > 0$.

The analysis is more involved for $\rho > 2\tau_\phi$, in which case the optimal $\tau_x^*$ may not be interior given the condition $\sigma_\varepsilon^2 > \underline{\Sigma}_C$ shown in Step 1. Therefore, we need to compare $\sigma_\varepsilon^2$ with $\underline{\Sigma}_I\left(\tau_\eta\right)$ to determine whether $\tau_x^*$ is interior. Differentiating $\underline{\Sigma}_I\left(\tau_\eta\right)$ with respect to $\tau_\eta$ yields

$$\frac{d\underline{\Sigma}_I\left(\tau_\eta\right)}{d\tau_\eta} = \frac{r^2}{4}\left[\frac{\left(\frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \tau_\eta\right) + 8\tau_\eta}{\sqrt{\left(\frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \tau_\eta\right)^2 + 8\tau_\eta^2}} - 1\right], \tag{A.38}$$

which is negative (positive) if $\tau_\eta$ is smaller (bigger) than $\frac{2}{9}\left(\tau_\phi - \frac{2}{\rho}\tau_\phi^2\right)$, and $\frac{2}{9}\left(\tau_\phi - \frac{2}{\rho}\tau_\phi^2\right)$ is positive given $\rho > 2\tau_\phi$. That is, $\underline{\Sigma}_I$ first decreases and then increases with $\tau_\eta$

(i.e., $\underline{\Sigma}_I\left(\tau_\eta\right)$ is U-shaped), and it achieves its minimum at $\underline{\Sigma}_I\left(\tau_\eta\right)|_{\tau_\eta=\frac{2}{9}\left(\tau_\phi-\frac{2}{\rho}\tau_\phi^2\right)} = \frac{4r^2}{9}\left(\tau_\phi-\frac{2}{\rho}\tau_\phi^2\right) = \underline{\Sigma}_C\left(\tau_\eta\right)|_{\tau_\eta=\frac{2}{9}\left(\tau_\phi-\frac{2}{\rho}\tau_\phi^2\right)}$. We complete the proof by analyzing two scenarios:

1. If $\sigma_\varepsilon^2 \leq \frac{4r^2}{9}\left(\tau_\phi-\frac{2}{\rho}\tau_\phi^2\right)$, we know $\sigma_\varepsilon^2 \leq \underline{\Sigma}_I\left(\tau_\eta\right)$ holds for $\forall\tau_\eta > 0$. In this case, $\tau_x^* = 0$ for all $\tau_\eta > 0$ and, hence, $\frac{d}{d\tau_\eta}\tau_x^* = 0$;
2. if $\sigma_\varepsilon^2 > \frac{4r^2}{9}\left(\tau_\phi-\frac{2}{\rho}\tau_\phi^2\right)$, the optimal $\tau_x^*$ can be interior. Specifically, $\sigma_\varepsilon^2 > \underline{\Sigma}_I\left(\tau_\eta\right)$ (hence, $\tau_x^* > 0$) if and only if $\tau_\eta \in \left(\tau_{\eta 1}\left(\sigma_\varepsilon^2\right), \tau_{\eta 2}\left(\sigma_\varepsilon^2\right)\right)$, where $\tau_{\eta 1}\left(\sigma_\varepsilon^2\right)$ and $\tau_{\eta 2}\left(\sigma_\varepsilon^2\right)$ are two real roots to equation $\underline{\Sigma}_I\left(\tau_\eta\right) = \sigma_\varepsilon^2$ such that $0 < \tau_{\eta 1}\left(\sigma_\varepsilon^2\right) < \frac{\sigma_\varepsilon^2}{2r^2} < \tau_{\eta 2}\left(\sigma_\varepsilon^2\right)$.[16] To verify $\frac{d}{d\tau_\eta}\tau_x^* \leq 0$ for $\tau_\eta < \frac{\sigma_\varepsilon^2}{2r^2}$, we analyze how the optimal $\tau_x^*$ changes as $\tau_\eta$ increases continuously from zero to $\frac{\sigma_\varepsilon^2}{2r^2}$. In particular, the optimal $\tau_x^*$ remains at $\tau_x^* \equiv 0$ for $0 < \tau_\eta \leq \tau_{\eta 1}\left(\sigma_\varepsilon^2\right)$; and becomes positive and monotonically increases in $\tau_\eta$ for $\tau_\eta \in \left(\tau_{\eta 1}\left(\sigma_\varepsilon^2\right), \frac{\sigma_\varepsilon^2}{2r^2}\right)$. The last part holds because both $\sigma_\varepsilon^2 > \underline{\Sigma}_I\left(\tau_\eta\right)$ and $\sigma_\varepsilon^2 > \underline{\Sigma}_C\left(\tau_\eta\right)$ are satisfied for $\tau_\eta \in \left(\tau_{\eta 1}\left(\sigma_\varepsilon^2\right), \frac{\sigma_\varepsilon^2}{2r^2}\right)$.

Inspecting the conditions in Part II, one can verify that $\tau_\eta < \frac{\sigma_\varepsilon^2}{2r^2}$ is a sufficient condition for $\frac{d}{d\tau_\eta}\tau_x^* \geq 0$, which completes the proof. ■

**Proof of Lemma 2:** Consider any investor $i$ who observes (i) the public signal $x$, (ii) the market price $p$, and (iii) has accumulated $N_i$ independent private signals $\{y_k = v+\eta_k\}_{k=1}^{N_i}$ via word-of-mouth communications prior to trading. Note that observing $N_i$ independently normally distributed signals is informationally equivalent to observing one signal $y_i = v + \varepsilon_i$, where $\varepsilon_i = \frac{\sum_{k=1}^{N_i}\eta_k}{N_i}$ and $\varepsilon_i \sim N\left(0, \frac{1}{N_i\tau_\eta}\right)$. Following the steps in the proof of Proposition 1, we can derive the equilibrium price and the equilibrium

[16] Note that given the condition $\sigma_\varepsilon^2 > \frac{4r^2}{9}\left(\tau_\phi-\frac{2}{\rho}\tau_\phi^2\right)$, we obtain $\frac{\sigma_\varepsilon^2}{2r^2} > \frac{2}{9}\left(\tau_\phi-\frac{2}{\rho}\tau_\phi^2\right)$ so $\underline{\Sigma}_I\left(\frac{\sigma_\varepsilon^2}{2r^2}\right) < \underline{\Sigma}_C\left(\frac{\sigma_\varepsilon^2}{2r^2}\right) = \sigma_\varepsilon^2$.

effort $a^*$. The result is the same as Proposition 1 except that we replace $\tau_\eta$ with $\bar{\mathbf{N}}\tau_\eta$, which is the mean precision of investors' private information. The amount of information investors learn from equilibrium price is now $\tau_p = \left(\bar{\mathbf{N}}\tau_\eta r\right)^2 \tau_\varepsilon$. It remains to show that $\bar{\mathbf{N}} = e^\lambda$. We know from Andrei and Cujean (2017, Proposition 1) that the average of the number of *incremental* signals collected by the investors prior to the trading is $e^\lambda - 1$. Therefore, $\bar{\mathbf{N}} = \left(e^\lambda - 1\right) + 1 = e^\lambda$ follows by adding the one private signal $y_i$ each investor is endowed with.

For the results in Proposition 2, 3, and 4, we have used the investors' residual uncertainty $\text{var}\left(v|\mathcal{F}\right)$ and their reliance $w_p$ on the price. Modeling private word-of-mouth communications will result in asymmetrically informed investors in terms of their private signal precisions. Therefore, different investors may face different levels of residual uncertainties and rely on price to a different extent. Nonetheless, we can construct a "representative investor" $R$ whose private information is as precise as the cross-sectional average precision among all the investors $\bar{\mathbf{N}}\tau_\eta$. We denote by $\mathcal{F}_R = \{p, x, y_R\}$ the information set of the representative investor $R$, where $p$ is the market price, $x$ is the firm's public disclosure, and $y_R$ is the private signal with a precision of $\bar{\mathbf{N}}\tau_\eta$. We can derive $R$'s residual uncertainty $\text{var}\left(v|\mathcal{F}_\mathcal{R}\right)$ and $R$'s reliance $w_p^R$ on the market price the same way we did in the main model. All derivations in Propositions 2, 3, and 4 then follow. ■

**Proof of Corollary 1:** By replacing $\tau_\eta$ with $\bar{\mathbf{N}}\tau_\eta$, we derive the optimal precision of public disclosure as follows:

$$\tau_x^* = 2\frac{1}{r^2\tau_\varepsilon} + \frac{2}{\rho}\tau_\phi^2 - \tau_\phi + \bar{\mathbf{N}}\tau_\eta - \left(\bar{\mathbf{N}}\tau_\eta r\right)^2 \tau_\varepsilon, \tag{A.39}$$

with $\tau_x^* > 0$ if and only if $\sigma_\varepsilon^2 = \frac{1}{\tau_\varepsilon} > \frac{r^2\left[\sqrt{\left(\frac{2}{\rho}\tau_\phi^2-\tau_\phi+\bar{\mathbf{N}}\tau_\eta\right)^2+8\left(\bar{\mathbf{N}}\tau_\eta\right)^2}-\left(\frac{2}{\rho}\tau_\phi^2-\tau_\phi+\bar{\mathbf{N}}\tau_\eta\right)\right]}{4}$. It is easy to show:

$$\frac{d\tau_x^*}{d\lambda} = \frac{d\tau_x^*}{d\bar{\mathbf{N}}}\frac{d\bar{\mathbf{N}}}{d\lambda} = \left(\tau_\eta - 2\bar{\mathbf{N}}\left(\tau_\eta r\right)^2\tau_\varepsilon\right)e^\lambda, \tag{A.40}$$

which is positive if and only if $\sigma_\varepsilon^2 = \frac{1}{\tau_\varepsilon} > \Sigma \doteq 2\bar{\mathbf{N}}\tau_\eta r^2$. ■

**Proof of Proposition 5:** It is easy to verify that $\Sigma = 2\bar{\mathbf{N}}\tau_\eta r^2$ in Corollary 1 increases with $r$ and $\tau_\eta$. ■

**Proof of Proposition 6:** Substituting $\tau_x^*$ characterized in Proposition 3 and $\tau_p = \left(r\bar{\mathbf{N}}\tau_\eta\right)^2\tau_\varepsilon$ into the linear pricing function, we can rewrite the coefficient $\alpha_\varepsilon$ as

$$\alpha_\varepsilon = \frac{1}{r\bar{\mathbf{N}}\tau_\eta}\frac{\bar{\mathbf{N}}\tau_\eta + \tau_p}{\tau_\phi + \tau_x^* + \bar{\mathbf{N}}\tau_\eta + \tau_p} = \frac{\frac{1}{r} + r\bar{\mathbf{N}}\tau_\eta\tau_\varepsilon}{2\left(\frac{1}{r^2\tau_\varepsilon} + \bar{\mathbf{N}}\tau_\eta\right) + \frac{2}{\rho}\tau_\phi^2}. \tag{A.41}$$

Straightforward calculation shows:

$$\frac{d}{d\lambda}\alpha_\varepsilon^{-1} = \frac{d\alpha_\varepsilon^{-1}}{d\bar{\mathbf{N}}}\frac{d\bar{\mathbf{N}}}{d\lambda} = -\frac{\frac{2}{\rho}r\tau_\eta\tau_\varepsilon}{\left(\frac{1}{r} + r\bar{\mathbf{N}}\tau_\eta\tau_\varepsilon\right)^2}e^\lambda < 0, \tag{A.42}$$

which proves the proposition. ■