Optimal Endowment Investing

Alexander Muermann
Vienna University of Economics and Business and VGSF*

Kent Smetters
Wharton and NBER*

August 25, 2018

Abstract

Is it really efficient for endowments to take substantial investment risk? Having "long horizons" isn't sufficient. More generally, why doesn't the Modigliani-Miller theorem apply to the relationship between endowments ("firms") and donors ("shareholders"), rendering an endowment's capital structure irrelevant? Under a condition—which quickly converges to standard DARA preferences in the number of donors—we show that risk taking reduces donor free-riding, is Pareto improving, and is required by competition among endowments for donations. Large endowments optimally take substantial risk even if donors are very averse to changes in endowment spending and expensive, risky investments don't outperform cash on average.

Keywords: Endowments, risk taking, charitable giving
JEL Code: G11

*Email addresses: alexander.muermann@wu.ac.at, smetters@wharton.upenn.edu. Yu Wang provided outstanding research assistance, including assistance with some of the proofs. We received helpful comments from Eduardo Azevedo, Zvi Bodie, Itay Goldstein, Jessie Handbury, Alex Rees-Jones, Judd Kessler, Nikolai Roussanov, and participants in the Wharton Applied Economics Seminar.

1 Introduction

Endowments take on substantial risk, including investments in equities, alternatives, and illiquid assets (e.g., timber).[1] Indeed, the so-called "endowment model" or "Yale model" of investment (Swensen (2009)) has become standardized, creating a blueprint for systematic endowment-like investing throughout the world (Leibowitz et al. (2010)).

There is a large existing literature on an endowment's optimal investment strategy, including hedging against economic downturns and random costs.[2] The conventional wisdom—tracing back to Litvack et al. (1974), Tobin (1974), Black (1976) and Swensen (2009)—is that an endowment can afford to take on substantial investment risk since it has a long time horizon, as much of its expenses occur in the future. Consistently, university endowments, for example, routinely emphasize their long-term investing horizon, with their main funds commonly labeled "long-term investment pools" or similar names.

At the same time, Black (1976) strongly argues that the standard argument of a long horizon is not well grounded because it takes an endowment in isolation of all of its stakeholders.[3] Neither his paper nor the subsequent literature, though, explores this point with sufficient micro-based foundations. The current optimal endowment literature does not incorporate explicit donor utility maximization, instead modeling the objective function of the endowment's sponsor (e.g., university) or some proxy for a range of stakeholders.

Commencing with the seminal 1958 theorem by Modigliani and Miller (MM), a literature in corporate finance began analyzing a firm's optimal capital structure by including its key stakeholders—the firm's shareholders—into the model. Barring financial market frictions (e.g., asymmetric information), the MM theorem shows that a firm's capital structure is irrelevant when shareholders are considered. Shareholders neutralize the firm's capital structure decision within their own private portfolios.

Similarly, an endowment does not operate independently of its donors. Donors give money to the endowment's sponsor (e.g., university) in order to receive some form of consumption value ("altruism" or "warm glow"). So, it might seem that the endowment investment problem could be relabeled as a traditional corporate finance problem, producing a MM type of theorem where the endowment's investment policy is irrelevant.

[1]See, for example, Lerner et al. (2008); Dimmock (2012); Cejnek et al. (2014a); and Ang et al. (2018).

[2]See, for example, Litvack et al. (1974); Tobin (1974); Black (1976); Merton (1992, 1993); Dybvig (1999); Fisman and Hubbard (2005); Swensen (2009); Constantinides (1993); Gilbert and Hrdlicka (2012); Cejnek et al. (2014a); Cejnek et al. (2014b); and, Brown and Tiu (2015).

[3]His argument is best understood using the more modern language of missing markets. Whereas the government can potentially take on more risk to complete missing markets between generations, an endowment lacks the necessary taxation authority. In particular, an endowment can't pre-commit future generations to donate during hard economic times.

The endowment problem, though, is different than the standard firm problem in one key way: the endowment's money comes from donors making voluntary gifts to a "public good" that produces *non-rivalrous consumption across donors* rather than a private return. In 1954, Paul Samuelson published his seminal (and second-most cited) article showing that the private sector will under-provide a public good relative to the "socially optimal" level, resulting in market failure.[4] The subsequent literature interpreted Samuelson's setting as "altruistic" agents donating to a public good, resulting in a "free-riding" (under-provision) in Nash equilibrium (e.g., Steinberg 1987; Andreoni 1988).

An endowment's sponsor raises money from donors, in large part, to support a common mission among its donors. This public good might include a passion for a university's sports teams, the shared joy of knowing that the university offers subsidies to low-income matriculates, the shared prestige of a university's contribution to basic research, or even a university's prestige as measured by the endowment size itself (James, 1990; Hansmann, 1990; Conti-Brown, 2011; Brown et al., 2014; Goetzmann and Oster, 2015; Chambers et al., 2015; Rosen and Sappington, 2016). Unlike a private good (e.g., a football ticket), each dollar gifted to these public goods produces non-rivalrous consumption simultaneously enjoyed by all donors. Free-riding in giving emerges because each donor receives utility value from the entire value of the public good even if she contributes little. Equivalently, each donor does not internalize the utility value to other donors from her own contribution. In still other words, each donor's contribution produces a *positive non-pecuniary "externality"* to other donors, resulting in a Prisoner's Dilemma.

If each donor internalized this externality, she would make a larger ("socially optimal") level of gift where all donors are better off, thereby achieving a Pareto improvement. The social optimum could be achieved using a Coasian contracting mechanism if there were few transaction costs to *centralizing* donor activity. It could also be achieved by force by a social planner imposing a head tax on each donor. This outcome would be "first best" because it would *directly* achieve the socially optimal level of gifts *without distortion*. However, first-best mechanisms are not plausible in our context. An endowment sponsor might, for example, engage in a capital campaign to inform donors of a desired target.[5] But, as with many public goods problems, transactions costs[6] or a lack of force requires the endowment's sponsor to rely on an *indirect*, "second-best," *decentralized* mechanism

[4]Standard examples of public goods include the shared benefit of non-congested roads and the military, the shared joy of knowing that low-income people are fed, and basic scientific research that produces non-pecuniary externalities.

[5]In fact, the target itself is endogenous and would likely be much larger if there were no free-riding problem. Put differently, the fact that capital campaigns often reach their initial target is not evidence against free-riding.

[6]In fact, even identifying "donors" would be subject free-riding at the "extensive" margin.

where some distortion, therefore, is required. This paper shows that the endowment's risk policy is such a mechanism, and a very powerful one.

Until now, a concern of free-riding has not been incorporated into the optimal endowment investment literature, as that literature has not considered explicit donor utility maximization.[7] However, maximizing an assumed endowment objective function without considering the donor problem raises an important question: How do you know that donors won't walk away? Donor utility maximization would be *required* with perfect competition across endowments for donor funds. Donor utility maximization is even required with *imperfect competition* but *complete information*, where donors oversee the endowment by, for example, serving on investment committees, as is common practice (Brown et al., 2011). To be sure, deviations from complete information are entirely plausible. But, even if the endowment environment has *asymmetric* information, donor maximization is needed to correctly identify principal-agent conflicts. For example, we show that the appearance of endowment excessive risk taking—commonly interpreted as an agency problem—is likely *efficient* and required by *complete* information. More generally, any claim of an agency issue must be made relative to the donors as a distinct principal.[8]

This paper, therefore, presents a micro-based model with donor utility maximization for determining an endowment's optimal capital structure. Our model is analogous to the approach of Modigliani and Miller's while incorporating Samuelson's insight since free-riding is part of the donor problem, which, as we show, makes the endowment investment policy very non-neutral. In particular, we first show that endowment risk taking reduces free-riding by generating equilibrium "precautionary donations" of *prudent* donors. (All reasonable preferences exhibit prudence.) This reduction in free-riding, though, comes at a cost of the endowment taking on risk (a second-best distortion) that it would not have taken if it were not for donor free-riding. We then derive an *optimality condition*—in which the absolute coefficient of prudence is sufficiently larger than the absolute coefficient of risk aversion—where it is, indeed, *optimal* for an endowment to take on this risk, that is, where the value of reducing free-riding exceeds the cost of additional risk. As the size of the donor base grows, this condition quickly converges to standard DARA preferences. Because our model maximizes a donor's expected utility, this additional endowment risk taking is Pareto improving and required by competition among endowments for donations, or, more generally, required by complete information.

[7]Some papers cited above incorporate an exogenous donation stream independent of the endowment's actions. The endowment does not maximize donor utility in these papers.

[8]If the endowment's problem is taken in isolation of the donor problem, there is no agency issue since the endowment is both principal and agent. Any observable investment pattern of investment by the endowment must simply reflect its objective function, which, by construction, must then be efficient.

In fact, we show that it is optimal for a large endowment to take on substantial risk even if donors are very risk averse toward variations in the endowment's value and even if an endowment uses high-free investments that, on net, pay no more than cash on average.[9] These results, therefore, contrasts sharply with "asset-liability matching" (or "liability-driven investing") that is appropriate with pension plans (Brown and Wilcox, 2009; Novy-Marx and Rauh, 2011), but where free-riding is *not* a concern.[10]

How does the presence of free-riding so radically change the optimal risk allocation? While we present a formal model in Section 2, it is worth considering a simplified example now to build some intuition. An endowment invest λ of its assets into the risky asset, with the remainder invested into a risk-free security paying a guaranteed zero return. The risky asset pays a zero expected return but with non-zero variance.[11] There are N identical donors, each endowed with one unit of wealth; hence, total wealth is also N. Each donor has separable log-log preferences: log utility over the post-return value of private assets that is added to log utility over the post-return value of the endowment. Given λ, donors play a public-goods Nash game that produces a symmetric equilibrium gift decision rule.[12] By backward induction, given this gift decision rule conditional on λ, the endowment then picks the second-best λ to maximize donor expected utility.

Suppose that the endowment initially fails to take risk, $\lambda = 0$. As we show in Section 5, with a two-point risk distribution, the Nash equilibrium produces an *aggregate* level of giving of $\frac{N}{N+1}$, which is less than the first-best socially optimal total level of giving of $\frac{N}{2}$, with two or more donors $N \geq 2$. However, if donors are prudent, the endowment can do better by taking on risk, $|\lambda| > 0$,[13] if the optimality condition, noted above, is satisfied. With log-log preferences, the optimality condition only requires having more than three donors, $N > 3$. Then, the endowment sets $|\lambda| > 0$, and the decentralized, second-best Nash equilibrium generates $\frac{N}{4}$ in total giving. Notice that with $N > 3$:

[9] Of course, we are not advocating the use of expensive investment products. Rather, our point is simply that substantial risk-taking, even with expensive investment products, is more efficient than no risk taking.

[10] Pension claims are paid by firms or governments as part of compensation. Neither party seeks voluntary donations to pay claims. Rather, these costs are financed from firm revenues or by taxation authority.

[11] Hence, the risky asset is second-order stochastically dominated by the risk-free asset. We make this assumption to focus on the mechanism at play, which, as we show, still produces positive risk taking. Obviously, if the risk premium were positive, the endowment is more likely to invest in risk.

[12] A public-goods game must be played since donors receive utility from the total endowment size, which includes gifts from other donors. If, instead, each donor *only* received utility from her own gift, this "warm glow" (Andreoni (1998)) is a private good without any shared benefit to other donors. Section 2 then shows that an equivalent MM theorem can emerge where the endowment investment policy λ has no impact on donor expected utility. In other words, the endowment investment policy is irrelevant (neutral) with only private goods. This neutrality also applies to previous papers where the endowment hedges random donation streams. Hence, without loss in generality, we can limit our consideration to donors who, as described above, derive at least *some* utility over the total value of the endowment.

[13] The model allows for shorting and so $\lambda \neq 0$ represents risk taking.

$$\frac{N}{N+1} \text{ (i.e., equilibrium total giving with } \lambda = 0)$$
$$< \frac{N}{4} \text{ (i.e., second-best equilibrium giving with } |\lambda| > 0)$$
$$< \frac{N}{2} \text{ (i.e., first-best ["social optimum"] giving)}$$

Consider the first inequality. It shows that endowment risk taking increases total equilibrium gifts, thereby reducing free-riding. Since donor expected utility is being maximized, this change is also Pareto improving and required by complete information. Of course, at first glance, this result seems absurd since the risky asset produces no commensurate risk premium. Put differently, this endowment is simply introducing *mean-preserving* risk. How could "junk variance" be optimal if donors are *risk averse* to changes in the endowment's value? The answer is that with positive *prudence*, even a mean-preserving endowment risk *effectively pre-commits* each donor to give a larger, precautionary donation in the decentralized Nash equilibrium. Under the optimality condition, the marginal benefit of reducing free-riding exceeds the marginal risk cost of additional risk, measured at the point where the endowment initially takes no risk ($\lambda = 0$).

However, endowment risk taking is a second-best mechanism and produces a level of giving less than first-best, as shown in the second inequality above. It is not efficient for the endowment to *fully* eliminate free-riding due to the distorting cost of risk-taking that does not exist in first best. (In first best, donors directly eliminate free-riding without changing the endowment's risk.) But, if the endowment tried setting λ to generate gifts more than the second-best value, donors would walk (or, would fire the management).

In sum, donors can't commit to a costless *centralized*, first-best Coasian mechanism. Endowment risk taking, pre-commits prudent donors to increase precautionary donations in the *decentralized* economy, but it comes with a (risk) distortion, making it second-best. We formally prove that the second-best utility can never be as high as the first-best.

The inequality above also shows that a *large* enough endowment ($N >> 3$) optimally invests almost exclusively in risk, even in this extreme setting with a zero risk premium. Notice that the first-best level of giving, $\frac{N}{2}$, grows *unbounded* in N. But, the level of giving with no endowment risk taking ($\lambda = 0$), $\frac{N}{N+1}$, converges to *unity*. Hence, the size of the free-riding problem—the difference between these two quantities—grows unbounded in N. A small endowment (i.e., small N) optimally takes no (if $N \leq 3$) to little risk ($N > 3$, but not big). But, a growing endowment must take on increasing risk to mitigate an exploding free-riding problem.

Now suppose donors are extremely averse to changes in the endowment's value. To be concrete, assume donors have log-CRRA(20) preferences: log utility over private assets that is added to Constant Relative Risk Aversion (CRRA) over the endowment's value but with an implausibly large risk aversion of 20.[14] The size of the free-riding problem, though, still grows in the number of donors N. Hence, for any *finite* level of risk aversion over the endowment's value, there exists a value of N where the optimality condition holds. The main difference relative to the log-log case is that the optimality condition now requires a larger value of N, now equal to about two dozen in the log-CRRA(20) case (Section 5). So, with just two dozen or more donors, an endowment optimally takes risk even if (i) it only invests in very expensive investment products paying a zero expected return and (ii) donors are very risk averse to changes in the endowment's value.

To be sure, these results might seem a bit far-fetched. However, recall that our model is only making two main assumptions. First, we assume donor maximization, consistent with competition, or, more generally, required by complete information, just like the MM theorem with respect to firm shareholders. Second, our model recognizes that our stakeholders are making investments into a public good that, in part, produces non-rivalrous consumption for other donors. Given standard assumptions that generate a unique Nash equilibrium, the optimal capital structure produced by our model is, therefore, the *unique solution* compatible with *complete information* and *altruistic donors*.

Of course, it is well known that the MM theorem fails in the face of real-world frictions, including asymmetric information. Yet, the MM theorem continues to be taught in introductory corporate finance and is one of the most cited papers in finance precisely because it tells us where to look for theoretically valid determinants of a firm's optimal capital structure. Similarly, we recognize that endowments face agency problems (Ehrenberg and Epifantseva, 2001; Core et al., 2006; Dimmock, 2012; Gilbert and Hrdlicka, 2013; Hoxby, 2015). Brown et al. (2014), for example, shows that the length of a university president's tenure is predictive of risk taking. Still, any agency issues are appropriately measured against the equilibrium with complete information.

Our results, therefore, reverse the conventional thinking about the relationship between endowment risk taking and agency. A *low* level of risk taking by a large endowment indicates a principal-agent conflict since this investment fails to maximize the expected utility of rational, fully-informed donors. A salaried endowment investment manager, for example, might take on a low level of risk for the same reason as a salaried CEO

[14] For example, proposed solutions to the classic "equity premium puzzle" generally try to reach a constant relative risk aversion of four or below. At a risk aversion of 20, it is doubtful that anyone who absorb the stresses of running a non-profit entity.

of a for-profit company: because his or her job represents a personalized, non-diversified risk. We are not, of course, ascribing intent, as a salaried endowment manager might unknowingly exploit the asymmetric information or bounded rationality of its donors. Executive stock options were created to increase CEO risk taking in for-profit companies (Hall and Murphy, 2002; Dittmann et al., 2017). Conversely, the backlash in recent years against university endowment managers receiving large performance-based payouts (e.g., Fleischer 2015) could actually *increase* this type of principal-agent problem.

The main focus of this paper is normative in nature, that is, on deriving *optimal* endowment risk taking rather than attempting to explain *actual* practice. Still, it is interesting that the model relationship between endowment size and optimal risk taking is largely consistent with the cross-sectional "size effect" found in the data, where risk taking increases in endowment size (NCSE (2017), Figure 3.2). However, contrary to conventional wisdom, we show that a large endowment should take on substantial risk even *without* the presence of fixed costs (e.g., dedicated asset managers) and even if it does not have access to unique asset classes that produce superior expected returns. It is also optimal for *smaller* endowments to take on less risk even in the presence of modern outsourced "endowment style" turnkey investment solutions that are supposed to absorb these fixed costs and provide access to superior returns.

The rest of this paper is organized as follows. Section 2 presents the model. Section 3 derives the socially optimal ("first best") endowment investment policy, consistent with a hypothetical social planner who directly resolves the free-rider problem by dictating the gift of each donor. Without the social planner construct, Section 4 derives the condition under which endowments optimally take on more risk in Nash equilibrium. It shows that this solution must be "second best" and can never deliver "first best" expected utility. Section 5 provides examples while Section 6 concludes. Appendix A contains proofs.

2 Three-Stage Game

For exposition purposes, we present a very simple model. An endowment is funded by N identical donors, each endowed with wealth $w = 1$. Both the endowment and donors have access to the same risky and risk-free assets. The risk-free asset pays a guaranteed zero real return, $r = 0$. The risky asset with random net return $\tilde{x}$ also has a zero expected value, $E[\tilde{x}] = 0$. The risk-free asset, therefore, second-order stochastically dominates the risky asset, and so risk-averse donors should never take on risk. We can interpret the risky asset as an expensive investment that does not over-perform cash on average. Nonetheless, we show that the endowment, which competitively maximizes ex-ante donor expected utility, should optimally hold the risky asset under a new condition we derive.[15]

The timing of the model is as follows:

Stage 1: The endowment announces its optimal investment policy in the risky asset λ that maximizes each donor's identical expected utility.

Stage 2: Each donor i picks her own optimal gift g_i. She simultaneously picks her own investment in risky assets α_i for the remainder of wealth not gifted, $w - g_i$. A Nash (non-cooperative) game is played with other donors.

Stage 3: The risky return $\tilde{x}$ is realized.

The endowment is forward looking and so the game is solved backward, starting with the decision-making at Stage 2.

2.1 Stage 2: Donors

Each donor i donates gift g_i to the endowment and also chooses the amount α_i to invest in risky assets from the remainder of her assets, $w - g_i$. Donor i makes these choices, conditional on the value of λ announced by the endowment at Stage 1 as well as the donation decisions of other donors, to maximize donor i's expected utility:[16]

$$EU_i\left(g_i, \alpha_i | \lambda, \overrightarrow{g_{-i}}\right) = E\left[u\left(1 + \alpha_i \tilde{x} - g_i\right)\right] + E\left[v\left(g_i + \sum_{j=1, j\neq i}^{N} g_j + \lambda \tilde{x}\right)\right]. \quad (1)$$

Here, $\overrightarrow{g_{-i}} = (g_1, ..., g_{i-1}, g_{i+1} ..., g_N)$ is the vector of donations made by donors other than donor i. (Consistently, we will use the notation $\vec{g} = (g_1, ..., g_N)$ to be the vector of all donations, including donor i.) The increasing and concave functions $u(\cdot)$ and $v(\cdot)$ provide felicity over personal consumption and gifts contained in the endowment, respectively.

The expression in function $v(\cdot)$ assumes that each donor only cares about the *sum* of gifts, consistent with pure altruism, and so the donor does not distinctively weigh her own gift. The endowment, therefore, provides a "public good" to donors in the tradition of Samuelson (1954), which can lead to free-riding.[17] A large subsequent literature has

[15] Numerical results are discussed below for the case $E[\tilde{x}] > r = 0$, consistent with a positive equity premium. The key results remain qualitatively unchanged but become quantitatively stronger.

[16] In particular, donor i invests α_i into the risky asset and the remainder, $(w - g_i - \alpha_i)$, into the risk-free asset, for a gross return at Stage 3 of $\alpha_i(1 + \tilde{x}) + (w - g_i - \alpha_i)(1 + r)$, which reduces to $1 + \alpha_i \tilde{x} - g_i$ with $w = 1$ and $r = 0$, as shown in the first term on the right-hand side of equation (1). Similarly, the endowment invests λ of total gifts, $G = \sum_{j=1}^{N} g_j$, into the risky asset and the remainder into the risk-free asset, to receive $\lambda(1 + \tilde{x}) + (G - \lambda)(1 + r) = G + \lambda \tilde{x}$, which is shown in the second term.

[17] One of the upshots of Samuelson (1954) is that government can use tax policy to reduce free-riding. A charitable deduction, in particular, could be incorporated in our model by reducing donor $i's$ private consumption by less than g_i in equation (1). However, it is generally inefficient for the government to

investigated actual giving motives to charities in general[18] and to education institutions, specifically.[19] In aggregate, the evidence suggests some form of "impure altruism," taking the form of a combination of private benefit (e.g., "warm glow" or, for example, increasing the chances of a child's admission to a college) and a "pure" benefit (altruism). For our purposes, however, we just need *some* altruism to motivate the free-riding problem.

2.2 Remark: A Neutrality Result with No Altruism

Indeed, suppose that, instead of altruism, each donor provides a gift that is perfectly substitutable with her own personal consumption. Then, the donor's problem (1) becomes

$$\begin{aligned} EU_i\left(g_i, \alpha_i; \lambda, \overrightarrow{g_{-i}}\right) &= E\left[u\left(1+\alpha_i\tilde{x} - g_i + (g_i - \lambda)(1+r) + \lambda(1+\tilde{x})\right)\right] \\ &= E\left[u\left(1+(\alpha+\lambda)\tilde{x}\right)\right], \end{aligned} \tag{2}$$

where, recall, $r = 0$. We drop the i subscript on α in the second equality since donors are still ex-ante identical and there is no public good. The free-riding problem, of course, vanishes. But, notice that each donor now only cares about the *total* risk, $(\alpha + \lambda)$, rather than its decomposition.[20]As a result, the actual endowment risk policy λ, that competitively maximizes donor ex-ante utility, is now irrelevant since each donor simply neutralizes any choice of λ with an offsetting choice of α, similar to the MM theorem. For the remainder of this paper, we consider the case of altruism giving shown in equation (1).

2.3 Stage 1: The Endowment

The endowment fund picks its investment policy λ to maximize the sum of donor utilities,

$$\sum_{i=1}^{N} E\left[u\left(1+\alpha_i^*(\lambda)\tilde{x} - g_i^*(\lambda)\right)\right] + N \cdot E\left[v\left(\sum_{i=1}^{N} g_i^*(\lambda) + \lambda\tilde{x}\right)\right],$$

where $g_i^*(\lambda)$ and $\alpha_i^*(\lambda)$ are the equilibrium policy functions that solve the Stage 2 prob-

fully resolve the free-riding problem, especially for large N, since the lost revenue must be replaced using distorting taxes. Incorporating the charitable deduction would not change our key qualitative results. Provided that the charitable deduction does not fully resolve the free-riding problem, endowments would still compete by increasing their investment in risk.

[18]See, for example, Becker (1976), Andreoni (1998), Fama and Jensen (1985), Rose-Ackerman (1996), and Fisman and Hubbard (2005).

[19]See, for example, Baade and Sundberg (1996), Clotfelter (2003), Ehrenberg and Smith (2003), Meer and Rosen (2009), Butcher et al. (2013), and Brown et al. (2015).

[20]This result easily generalizes to the case $E[\tilde{x}] > r = 0$ and other potential complexities.

lem.[21] However, since donors are ex-ante identical, maximizing their sum of utilities is identical to choosing λ to maximize the ex-ante utility of a single donor, consistent with a competitive equilibrium, where endowments compete for donations:

$$\Omega(\lambda) = E\left[u\left(1 + \alpha^*(\lambda)\tilde{x} - g^*(\lambda)\right)\right] + E\left[v\left(Ng^*(\lambda) + \lambda\tilde{x}\right)\right]. \tag{3}$$

3 Optimal Endowment Investment: Social Optimum

The endowment problem that maximizes equation (3) looks similar to the "social planner" problem considered by Samuelson (1954) and the large subsequent literature on the private provision of a public good. However, there is a subtle but important difference.

While picking λ, the social planner also picks the gifts vector $\vec{g}$ when maximizing equation (3). The social planner, therefore, directly solves the gifts free-riding problem, producing the *first-best* expected donor utility. Let hatted variables corresponding to the social planner problem, and variables with $*$ superscripts denote optimal values.

Theorem 1. *In the social optimum,* $\widehat{\lambda}^* = 0$, $\widehat{\alpha}_i^* = 0\ \forall i$, *with equals gifts* $\widehat{g}_i^* = \widehat{g}^*$ *that solves*

$$u'\left(1 - \widehat{g}^*\right) = Nv'\left(N\widehat{g}^*\right). \tag{4}$$

Intuitively, since the risky asset is dominated, the endowment and individual donors don't take on any risk. Since the social planner directly controls individual donor gifts, he does not need to inefficiently distort risk taking by setting $\widehat{\lambda}^*$ to any value but zero. The concavity of $u(\cdot)$ and $v(\cdot)$ implies that this solution is also unique.

4 Optimal Endowment Investment: Nash Equilibrium

In contrast, an endowment investment manager cannot pick gifts. Instead, he can only control the endowment's investment policy λ. As we prove below, risk taking by the endowment ($\lambda \neq 0$) increases gifts of prudent donors, decreases free-riding of prudent donors. But this reduction in free-riding comes at a cost of more risk taking. We derive a new condition where the benefit of reducing free-riding exceeds the cost of risk taking.

[21] Section 4 defines the non-cooperative (Nash) equilibrium in more detail, but it is standard. For the cooperative (social) case considered below, the hypothetical social planner directly picks the gift "policy functions."

Relative to the social planner, who directly solves free-riding by dictating gifts and not distorting investments, picking λ is, therefore, a *second-best* mechanism.

We now solve for the non-cooperative solution (Nash) for the game outlined in Section 2 and then compare it against the social optimum solution without free-riding.

4.1 Stage 2: Nash Equilibrium Gifts

Starting first with the donor game in Stage 2, the definition of a Nash equilibrium is standard. Each donor i picks the tuple (g_i, α_i) that maximizes her problem (1), given the gifts made by other donors, $\overrightarrow{g_{-i}}$.[22] A Nash equilibrium is the vector of gifts $\vec{g^*}$ and the vector of personal risk taking $\vec{\alpha^*} = (\alpha_1, ..., \alpha_N)$ that maximizes the donor problem (1), $\forall i$.

Theorem 2. *The Nash equilibrium in the Stage-2 donor game is unique with $\alpha_i^* = 0$, $\forall i$, and equal gifts, $g_i^*(\lambda) = g^*(\lambda)$, conditional on λ, that solves*

$$u'(1 - g^*(\lambda)) = E\left[v'(Ng^*(\lambda) + \lambda\tilde{x})\right]. \tag{5}$$

In words, each donor wants to invest all of her personal (non-gifted) wealth, $w - g_i$, into the risk-free asset, which, recall, pays the same expected return as the risky asset. The equilibrium is unique and symmetric, producing identical gifting policy functions, $g^*(\lambda)$.

4.2 Stage 2: Comparative Statics of $g^*(\lambda)$

We now investigate how the equilibrium level of giving, $g^*(\lambda)$, changes endowment risk away from its socially optimal value, $\widehat{\lambda}^* = 0$. Let $P^v(\cdot) \equiv -\frac{v'''(\cdot)}{v''(\cdot)}$ denote the Arrow-Pratt coefficients of absolute prudence for felicity function v over the public good. Then:

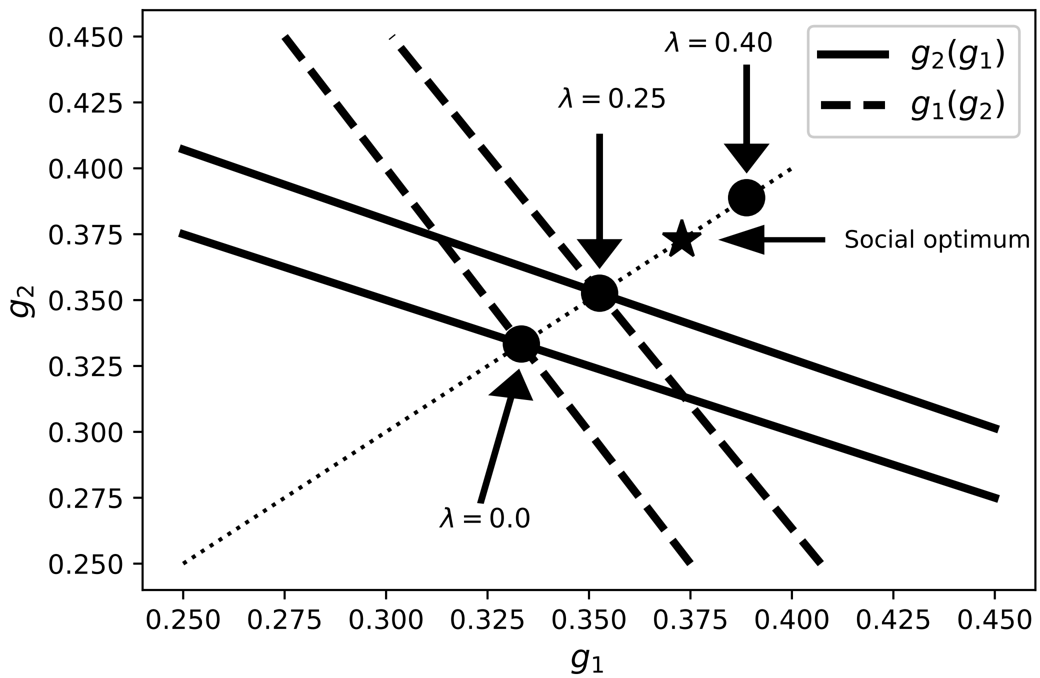
Theorem 3. $P^v(Ng^*(0)) > 0 \Longleftrightarrow \lambda = 0$ *is a local* <u>*minimum*</u> *of* $g^*(\lambda)$.

In words, with positive prudence, some endowment risk taking will increase the Nash equilibrium level of giving relative to the socially-optimal level of risk taking $\widehat{\lambda}^* = 0$.[23]

[22] Each donor only needs to know the total size of giving, which is standard in public good games. Even the Nash equilibrium concept can be replaced with a Perfect Bayesian Equilibrium under reasonable conditions, but this modification isn't necessary since total gifts are generally observable. For example, a university typically provides updated information about gifts during a capital campaign.

[23] The statement in Theorem 3 is sufficient for our purposes. Additional statements, which are proven in the proof for Theorem 3 can be made. In particular, $P^v(\cdot) > 0 \Longrightarrow \lambda = 0$ is a *global* minimum of $g^*(\lambda)$. Moreover, $g^*(\lambda)$ is monotonically decreasing when $\lambda < 0$ and monotonically increasing for when $\lambda > 0$.

Figure 1: Gifts Reaction Functions for Two Donors ($N = 2$)



Notes: Shows reaction functions and corresponding equilibria for two donors $i \in \{1, 2\}$, $\lambda \in \{0.0, 0.25, 0.40\}$, $u(w) = v(w) = \frac{w^{1-\gamma}}{1-\gamma}$, $\gamma = 4$, and $\alpha_i = 0$. Net return to stocks, $\tilde{x}$, take values in set $\{-1., -0.05, 0.05, 1.0\}$ with equal probability. At $\lambda = 0$, the reaction functions are linear since $(w_0 - g_i) = (g_1 + g_2)$ at optimum. For $\lambda > 0$, reaction functions are slightly nonlinear.

The role of prudence can be explained with equation (5). As the endowment changes λ from zero, it introduces risk in the second term, $E[v(\cdot)]$. Donor i makes a "precautionary donation" with a larger gift, g_i, similar to "precautionary savings" with uninsurable risk (Kimball, 1990). Positive prudence is a standard assumption. DARA preferences, for example, is a sufficient (but not necessary) condition for positive prudence.

Consider the case of two donors ($N = 2$), Figure 1 plots the gift *reaction* functions of donors $i = \{1, 2\}$ in gift (g_1, g_2) space with $\alpha_i = 0$. Donors have constant relative risk averse felicity (see Figure 1 notes for more details). In the set of reaction functions, labeled as "$\lambda = 0.0$," the value λ is set to zero and the intersection of the reaction functions represents the Nash equilibrium level of gifting. (By symmetry, $g_1^* = g_2^*$, all equilibria lie on the dotted 45-degree line.) The equilibrium gift level lies below the indicated social optimum value. For the set of gift reaction functions with $\lambda = 0.25$, equilibrium gifts move closer to the social optimum, helping reduce some free-riding.

4.3 Stage 1: Optimal λ^*

But, is it actually *efficient* for the endowment to take on risk? Without free-riding, Theorem 1 shows that the optimal endowment level of risk taking is zero, $\widehat{\lambda}^* = 0$. With free-riding, Theorem 3 shows that endowment risk taking reduces free-riding. So, under what condition does the benefit to donors from less free-riding exceed the cost from risk taking?

In Stage 1, the endowment solves for its optimal investment policy λ^* by maximizing problem (3), given the Nash gift policy functions $g^*(\lambda)$ determined in Stage 2. Notice that *if* the endowment fails to take on any risk ($\lambda = 0$) then Nash gifts are implicitly determined simply by the relationship,

$$u'(1 - g^*) = v'(Ng^*) . \tag{6}$$

Compare this relationship to equation (4) in Theorem 1 that derives the social optimal level of giving. The only difference is the presence of N that multiples the marginal utility of the public good on the right-hand side in equation (4). Intuitively, since the public good (endowment) is non-rivalrous in consumption, the socially optimal solution linearly increases the marginal utility of the public good. It is easy to see that $g^*(\lambda = 0) < \widehat{g}_i^*$, the now-familiar public goods free-riding problem first identified by Samuelson (1954).[24]

For exposition, suppose $u(\cdot) = v(\cdot)$. (Footnotes and Appendix A generalize the results to $u(\cdot) \neq v(\cdot)$). Denote $A(\cdot) \equiv -\frac{u''(\cdot)}{u'(\cdot)}$ and $P(\cdot) \equiv -\frac{u'''(\cdot)}{u''(\cdot)}$ as the Arrow-Pratt coefficients of absolute risk aversion and absolute prudence, respectively. Recall each donor has wealth $w = 1$ before making a gift, and so the donor count N equals total wealth.

Theorem 4. $\lambda = 0$ *is a local* <u>*minimum*</u> *solution to the endowment problem that maximizes equation (3)* $\Longleftrightarrow P\left(\frac{N}{N+1}\right) > \frac{N+1}{N-1} \cdot A\left(\frac{N}{N+1}\right)$ *when* $u(\cdot) = v(\cdot)$.[25] *Hence,* $|\lambda^*| > \widehat{\lambda}^* = 0$.

Theorem 4 shows that the endowment's optimal investment policy λ^* *is* in Nash equilibrium is to take on risk (by longing or shorting the risky asset[26]), in fact, by more than is socially optimal $\widehat{\lambda}^*$, under a new condition where the felicity's level of absolute pru-

[24]Indeed, the expression shown in Theorem 1 is commonly known as the "Samuelson condition."

[25] As shown in the proof to Theorem 4 in Appendix A, for the general case where $u(\cdot)$ and $v(\cdot)$ might be different, the necessary and sufficient condition is:

$$(N-1) P^v (Ng^*(0)) > A^u (1 - g^*(0)) + NA^v (Ng^*(0)) ,$$

where P^v is the absolute prudence of v. A^u and A^v are the absolute risk aversions for u and v, respectively. We focus on the equality case in the text for exposition, thereby dropping the superscripts on P and A.

[26]Our characterization of risky returns does not distinguish between long or short positions. Any deviation from zero represents risk taking.

dence is sufficiently larger than its absolute risk aversion. For a finite value of N, this condition is slightly stronger than the relationship between prudence and risk aversion that is equivalent to standard Decreasing Absolute Risk Aversion (DARA) preferences, i.e., $P\left(\frac{N}{N+1}\right) > A\left(\frac{N}{N+1}\right)$.[27] However, notice that this new condition quickly converges to DARA in N since the multiplier, $\frac{N+1}{N-1}$, in the Theorem-4 condition, converges to 1.

To understand this relationship, first consider the term $\frac{N}{N+1}$ inside of the $P(\cdot)$ and $A(\cdot)$ operators in Theorem 4. At $\lambda = 0$, equation (6) with $u(\cdot) = v(\cdot)$ implies that the individual Nash equilibrium gift $g^*(0) = \frac{1}{N+1}$. Hence, total gifts equal $\frac{N}{N+1}$ at $\lambda = 0$.

At large N, therefore, the Theorem-4 condition simply requires DARA preferences, $P\left(\frac{N}{N+1}\right) > A\left(\frac{N}{N+1}\right)$, at the total levels of gifts with no endowment risk taking, $\lambda = 0$. Intuitively, as shown in the Theorem 3, positive prudence, $P(\cdot) > 0$, implies that endowment risk taking, $|\lambda^*| > 0$, reduces free-riding, thereby capturing the marginal benefit of additional risk taking. However, as shown in Theorem 1 risk taking is not optimal without free-riding. The coefficient of relative risk aversion, $A(\cdot)$, captures the marginal cost of additional risk. It is optimal, therefore, for the endowment to take on risk if the marginal benefit, $P\left(\frac{N}{N+1}\right)$, exceeds the marginal cost, $A\left(\frac{N}{N+1}\right)$, calculated at the level of gifts with no risk taking.

The results generalize to the case where N is not large and $u(\cdot) \neq v(\cdot)$, although with some additional notation.[28] Simulation code, to be made available online, considers the case of $E[\tilde{x}] > r = 0$, consistent with a positive equity premium.[29]

[27] DARA is "a very intuitive condition" (Eeckhoudt et al., 2005) and includes commonly-used preferences such as Constant Relative Risk Aversion. DARA is necessary and sufficient for the absolute amount of risk taking to increase in wealth, along with many other standard properties (Gollier, 2004).

[28] At smaller N, the multiplier, $\frac{N+1}{N-1}$, is more relevant. Consider the more general Theorem-4 condition from footnote 25, where we multiply each side by $\frac{1}{2}E\left[\tilde{x}^2\right]$:

$$(N-1)\,\frac{1}{2}E[\tilde{x}^2]P^v\left(Ng^*(0)\right) > \frac{1}{2}E[\tilde{x}^2]A^u\left(1-g^*(0)\right) + N\frac{1}{2}E[\tilde{x}^2]A^v\left(Ng^*(0)\right).$$

The left-hand side, which is the Arrow-Pratt approximation for the precautionary equivalent premium (Kimball, 1990), is exactly equal to the willingness of $N-1$ donors to increase their precautionary donations in response to additional risk taking by the endowment in Nash equilibrium, i.e., it represents the benefit of additional risk taking to an individual donor. The right-hand side, which is the Arrow-Pratt approximation for the risk premium, is exactly equal to the premium required by an individual donor to take on the additional risk, i.e., the cost of additional risk taking. More specifically, the right-hand side is equal to the direct cost associated with more risk taking by the endowment as a whole, $N\frac{1}{2}E[\tilde{x}^2]A^v\left(Ng^*(0)\right)$, and in the donor's "private wealth", $\frac{1}{2}E[\tilde{x}^2]A^u\left(1-g^*(0)\right)$. Even though $\alpha^* = 0$, the envelope theorem implies that, in effect, a donor's private wealth, up to the individual gift level, $g^*(0)$, is exposed to additional risk.

[29] To summarize, the key results presented herein remain qualitatively unchanged but generally become quantitatively stronger. The reason is that *first* term in equation (5) now contains some risk since $\alpha^* > 0$. This risk creates a competing prudence effect for the endowment, which must increase λ even more to offset.

4.4 Comparison Against the Social Equilibrium

The case of $\lambda = 0.40$ in Figure 1 shows that the Nash equilibrium level of gifts can overshoot the social optimum level of gifts. Indeed, there exists a value of λ that fully eliminates the free-riding problem. Importantly, however, this value of λ is generally *not* second-best optimal, as it does not maximize the Stage-1 endowment problem (3).

Intuitively, *fully* solving the gifts free-riding problem in the Nash game would require distorting the endowment's second-best investment policy instrument λ too much. In fact, for the example shown in Figure 1, where $N = 2$ and $\gamma = 4$, the second-best level of endowment risk taking, λ^*, is 0! As we show in Section 5, with $\gamma = 4$, there must be at least 10 donors ($N \geq 10$) for the necessary and sufficient Theorem-4 condition to be satisfied with HARA utility (of which Constant Relative Risk Aversion is a special case).

Now assume $N = 10$. Figure 2 plots the value of expected utility $\Omega(\lambda)$ from equation (3), where the equilibrium Stage-1 policy functions $\alpha^*(\lambda)$ and $g^*(\lambda)$ are Nash. Notice that the expected utility peaks at a value λ greater than zero but falls at larger values of λ. Increasing λ above zero reduces free-riding, thereby increasing donor expected utility. But, raising λ too much, reduces donor utility by distorting risk taking too much.

In sharp contrast, Figure 3 shows the donor expected utility for the same donor problem where the social planner can directly pick gifts. Notice that expected utility now peaks at $\lambda = 0$, consistent with Theorem 1. In the first-best setting, there is no need to take on otherwise inefficient risk taking since the free-riding problem can be directly solved.

More generally, the Nash (second-best) expected utility, calculated at the optimal endowment investment policy $\Omega(\lambda^*)$ with $|\lambda^*| > 0$, can never produce the socially-optimal (first-best) level of expected utility. At only a slight abuse of notation, denote $\Omega_{SO}(\lambda)$ as as the value of expected utility $\Omega(\lambda)$ in equation (3), but where the social planner picks λ *and* individual gifts in Stage 1. Then:

Theorem 5. *Under the Theorem-4 condition,* $\Omega_{SO}\left(\widehat{\lambda}^* = 0\right) > \Omega(\lambda^*) > \Omega(\lambda = 0)$, *where* $|\lambda^*| > 0$.

The first inequality shows that the socially optimum solution, where social planner directly picks gifts and the endowment takes no risk, produces larger donor expected utility than the second-best Nash optimum where the only available instrument is for the endowment to take on investment risk. (This inequality does not require the Theorem-4 condition.) Once in the second-best setting, the second inequality shows that positive risk taking is optimal under the Theorem-4 condition, as previously proven in Theorem 4.

Figure 2: Expected Utility: Nash

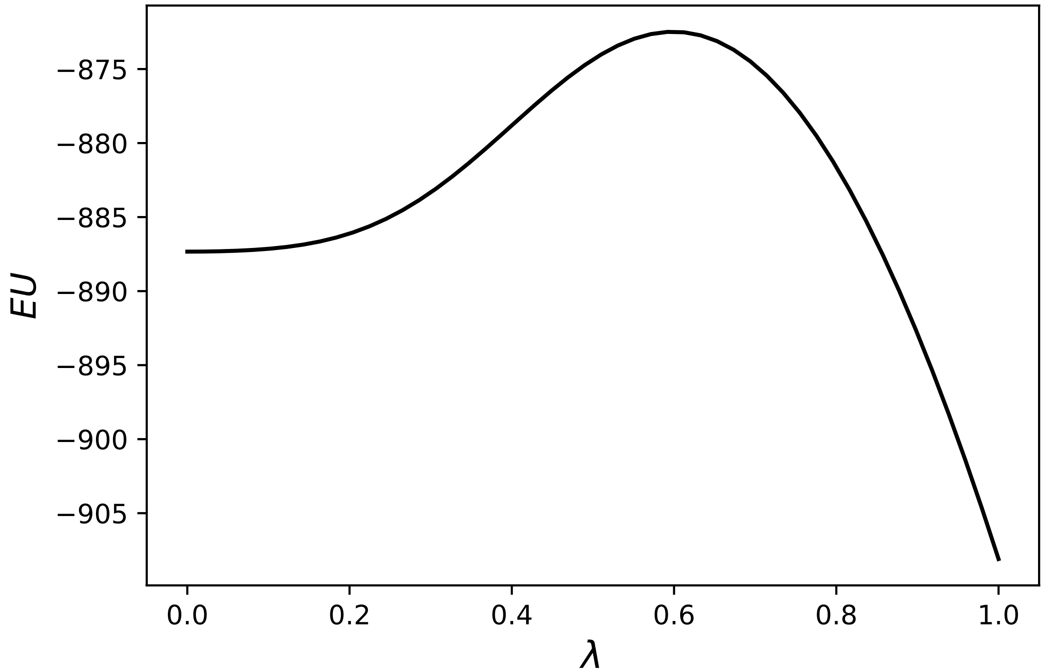
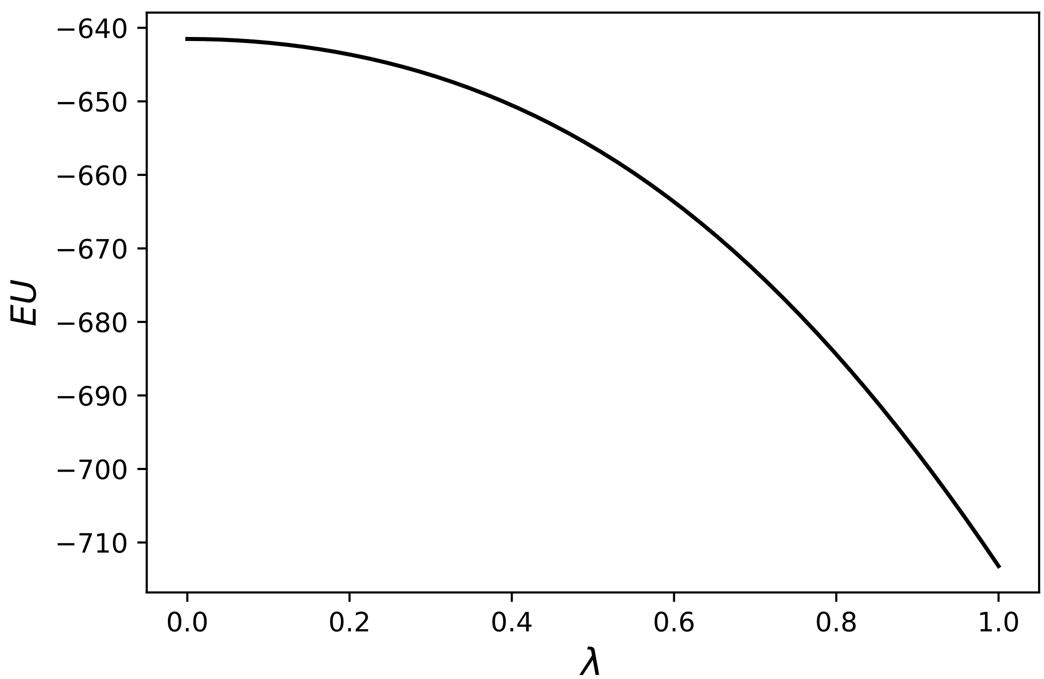
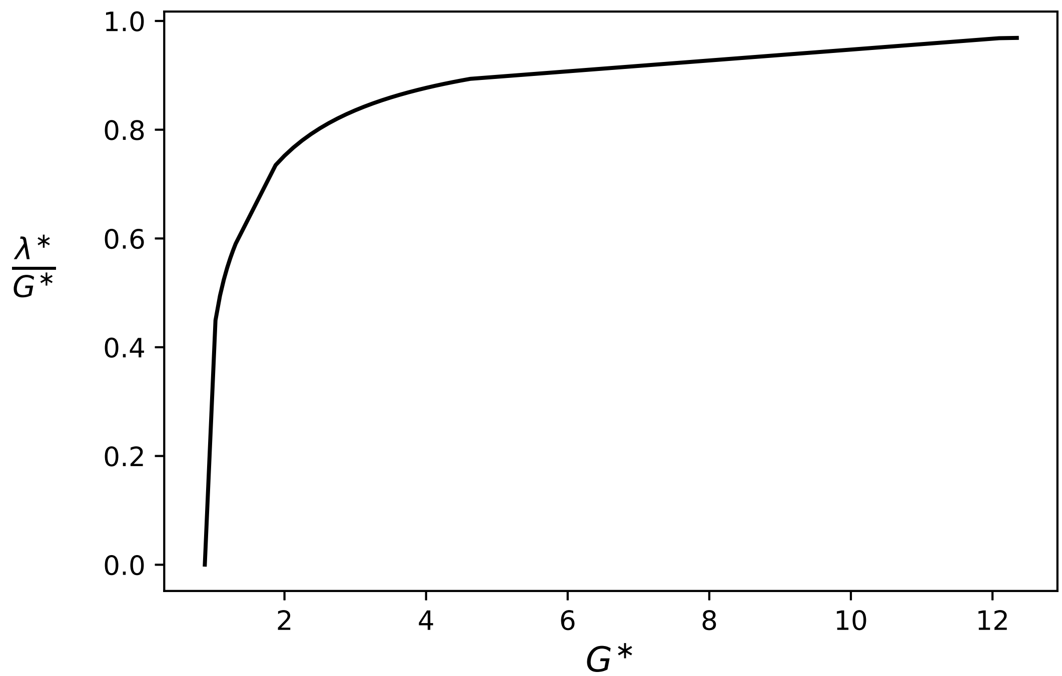


Figure 3: Expected Utility: Social Optimum



Notes for Figures 2 and 3: $N = 10$, $u(w) = v(w) = \frac{w^{1-\gamma}}{1-\gamma}$, and $\gamma = 4$. Net return to stocks, $\tilde{x}$, take values in set $\{-1., -0.05, 0.05, 1.0\}$ with equal probability.

Figure 4: Investment in Risky Asset Relative to Endowment Size



Notes: $u(w) = v(w) = \frac{w^{1-\gamma}}{1-\gamma}$, and $\gamma = 4$. Net return to stocks, $\tilde{x}$, take values in set $\{-1., -0.05, 0.05, 1.0\}$ with equal probability. The value of G^* increases by increasing N.

4.5 A First-Order, Size Effect

Denote $G^*(\lambda) = \vec{g}(\lambda) \cdot \vec{1}$ as sum of all gifts in Nash equilibrium, equal to the size of the total endowment. Then, the ratio $\frac{\lambda^*}{G(\lambda^*)}$ gives the *optimal share* of the total endowment invested into risky assets. Figure 4 shows that the optimal share increases in the size of the endowment, G, with the optimal investment share eventually approaching 100%. Larger endowments optimally take on substantial risk taking even without fixed costs in asset management or access to superior expected returns.[30]

Intuitively, this "size effect" is required to reduce free-riding by each donor i in the presence of the sum of other gifts, $G^*_{-i}(\lambda) = \vec{g_{-i}}(\lambda) \cdot \vec{1}$. In the Nash game in Stage 2, each donor i takes G_{-i} as given. A larger G_{-i} creates a "buffer stock" to donor i, reducing the prudence effect at any given value of λ. As the endowment size G increases, it is optimal to increase λ faster than G in Stage 1 to recovery the lost prudence effect to each donor.[31]

[30] Section 5 provides analytic examples. Under positive analysis, the shape of the curve in Figure 4 could be calibrated by picking $v(\cdot)$ as a weighted value of $u(\cdot)$, or giving $v(\cdot)$ its own risk aversion.

[31] Any "existing assets" from previous periods are identical to donor i facing a large buffer stock G_{-i}.

5 Examples

We now present three examples, where $u(\cdot) = v(\cdot)$, starting with very general HARA preferences before narrowing.

5.1 HARA Utility

Consider the HARA class of felicity functions,

$$u(w) = \zeta \left(\eta + \frac{w}{\gamma}\right)^{1-\gamma},$$

on the domain $\eta + \frac{w}{\gamma} > 0$. The first three derivatives are:

$$\begin{aligned} u'(w) &= \zeta \frac{1-\gamma}{\gamma} \left(\eta + \frac{w}{\gamma}\right)^{-\gamma}, \\ u''(w) &= -\zeta \frac{1-\gamma}{\gamma} \left(\eta + \frac{w}{\gamma}\right)^{-\gamma-1}, \text{ and,} \\ u'''(w) &= \zeta \frac{(1-\gamma)(1+\gamma)}{\gamma^2} \left(\eta + \frac{w}{\gamma}\right)^{-\gamma-2}. \end{aligned}$$

We naturally assume $\zeta \frac{1-\gamma}{\gamma} > 0$ such that $u'(w) > 0$ and $u''(w) < 0$. The coefficients of absolute risk aversion and prudence are given by

$$\begin{aligned} A(w) &= \left(\eta + \frac{w}{\gamma}\right)^{-1}, \text{ and,} \\ P(w) &= \frac{1+\gamma}{\gamma} \left(\eta + \frac{w}{\gamma}\right)^{-1}, \end{aligned}$$

respectively. Then the Theorem-4 condition,

$$P\left(\frac{N}{N+1}\right) > \frac{N+1}{N-1} \cdot A\left(\frac{N}{N+1}\right),$$

is equivalent to

$$0 < \gamma < \frac{N-1}{2}. \tag{7}$$

For example, with the value $\gamma = 4$, as used in the previous examples, it takes just 10 donors for it to be optimal for the endowment to take on investment risk, $|\lambda^*| > 0$. Even smaller values of N are required at smaller values of γ (less concavity). At large N, this condition simply becomes $0 < \gamma < \infty$, which is equivalent to DARA preferences.

For the subset of constant relative risk aversion (CRRA) felicity functions ($\eta = 0$), when $u(\cdot)$ and $v(\cdot)$ might be different with relative risk aversion parameters γ_u and γ_v, equation (7) is replaced by:

$$0 < \gamma_u + \gamma_v < N - 1 \tag{8}$$

$$0 < \gamma_v. \tag{9}$$

Equations (8) and (9) indicate that the more linear the preferences, the lower the required hurdle value for N. These conditions support very flexible preferences.

Remark (Quasi-linear utility)**.** *Preferences can be quasi-linear. In the extreme, preferences over non-charitable consumption can be linear, $\gamma_u = 0$, and preferences over charitable consumption can be arbitrarily close to linear, $\gamma_v \to 0^+$. If $\gamma_v = 0$ then it would be socially optimal for each donor to donate <u>all</u> of her wealth above a small non-charitable value.*

The model also accommodates so-called "deterministic" planning goals.

Remark ("Deterministic" future expenses)**.** *If the endowment's sponsor's future spending goals are fully fixed ("deterministic"), then $\gamma_v = \infty$. In this case, condition (8) cannot hold for any finite value of N, and so the optimal endowment risk is zero, $\lambda^* = 0$. However, $\gamma_v = \infty$ also implies that non-charitable consumption, $1 - g_i$, converges to zero even in Nash equilibrium. More realistically, suppose that future spending goals by the endowment's sponsor are at least somewhat flexible. Then, the value of γ_v is finite and there exists a finite value of N where endowment risk taking is optimal, $|\lambda^*| > 0$. For example, suppose $\gamma_u = 4$ and $\gamma_v = 20$, suggesting high risk aversion to the endowment's asset value falling short of expectation. Then, an endowment with $N \leq 25$ donors optimally takes no risk while an endowment with $N \geq 26$ does.*

5.2 Log Utility with a Two-Point Risk Distribution

Consider the subset of the HARA felicity set with constant relative risk aversion (CRRA):

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma}.$$

Then $u'(w) = w^{-\gamma}$. The social optimally gift level without endowment risk taking is:[32]

$$\widehat{g}^*\left(\widehat{\lambda}^* = 0\right) = \frac{N^{1/\gamma}}{N^{1/\gamma} + N}$$
$$= \frac{1}{2}, \text{ where } \gamma = 1 \text{ (i.e., log utility).}$$

As noted, the second equality represents the special case of log utility, $u(w) = \ln(w)$.

To compute the corresponding Nash equilibrium, we focus on the case of log utility ($\gamma = 1$) in the remainder of this example. Then, the Theorem-4 condition,

$$P\left(\frac{N}{N+1}\right) > \frac{N+1}{N-1} \cdot A\left(\frac{N}{N+1}\right),$$

is satisfied for all $N > 3$. Furthermore, to obtain analytic solutions for the Nash game, suppose $\tilde{x}$ follows a two-point distribution that takes the values $+1$ and -1 with equal probability, $\frac{1}{2}$.[33] Then, the Nash equilibrium donation is

$$g^*(\lambda) = \frac{N + \sqrt{N^2 + 4N(N+1)\lambda^2}}{2N(N+1)}. \tag{10}$$

Equation 10 implies that *if* λ were set at zero (no endowment risk taking), then $g^*(\lambda = 0) = \frac{1}{N+1}$, which is less than the social optimal value of $\frac{1}{2}$, with $N > 3$. Importantly, individual Nash gifts converge to zero in N while total gifts $G^* = Ng^*(0)$ converges to just unity. At Stage 1, the endowment, therefore, optimally chooses λ to grow with N:

Theorem 6. *Suppose felicity is log and let $\tilde{x}$ follow a two-point distribution that takes the values $+1$ and -1 with equal probability. Then, $|\lambda^*| = \frac{\sqrt{N(N-3)}}{4}$ and $g^*(\lambda^*) = \frac{1}{4}$.*

Notice that individual gift giving now stays constant at $\frac{1}{4}$. (Total gift giving, therefore, rises linearly with N.) Of course, this gift level is below the socially optimal level, $\frac{1}{2}$, derived above. In the Nash game, setting λ at a larger value than shown in Theorem 6 would generate generate larger gifts but at the cost of too much distortion to λ.

[32]For HARA, the socially optimal gift (with $\widehat{\lambda}^* = 0$) is

$$\widehat{g}^* = \frac{(N)^{1/\gamma} + \left((N)^{1/\gamma} - 1\right)\gamma\eta}{(N)^{1/\gamma} + N}.$$

[33]For the socially optimum, the risk distribution is irrelevant since $\widehat{\lambda}^* = 0$.

Moreover, the optimal endowment share invested in risky assets is equal to

$$\frac{|\lambda^*|}{G^*(\lambda^*)} = \frac{|\lambda^*|}{Ng^*(\lambda^*)} = \frac{\sqrt{N(N-3)}/4}{N/4} = \sqrt{1-3/N}.$$

At $N = 3$, the Theorem-4 condition is violated, and so the endowment takes no risk. For $N > 3$, optimal risk-taking increases in N, converging to 1 in endowment size ($G^* = \frac{N}{4}$).

5.3 CARA Utility

Our final example tests the boundary of the Theorem-4 condition, which, recall, is sufficient *and* necessary for the endowment to take on risk in the Nash game, $|\lambda^*| > 0$.

Consider the case of Constant Absolute Risk Aversion (CARA) felicity:

$$u(w) = -e^{-\gamma w}$$

The socially optimal level of individual gifts are:

$$\widehat{g}^*(\widehat{\lambda}^* = 0) = \frac{1}{N+1}\left(1 + \frac{\ln N}{\gamma}\right).$$

For the Nash game, to obtain an analytic solution, assume that $\tilde{x}$ is normally distributed with expectation 0 and and variance σ^2. The Nash gift policy function is

$$g^*(\lambda) = \frac{1}{N+1}\left(1 + \frac{1}{2}\gamma\sigma^2\lambda^2\right). \tag{11}$$

Like the CRRA example considered above, *if* λ were set to zero, then individual gifts $g^*(\lambda = 0)$ fall to zero in N and the value of the total gift, Ng^*, approaches the value of just unity. Moreover, individual gifts are below the socially optimal level of gifts.

However, CARA presents a problem not previously found with CRRA. In particular, CARA felicity implies that $P\left(\frac{N}{N+1}\right) = \gamma = A\left(\frac{N}{N+1}\right) < \frac{N+1}{N-1} \cdot A\left(\frac{N}{N+1}\right)$. Hence, CARA violates the Theorem-4 condition, even if "just barely" at large N. As a result, in Stage 1, it is no longer optimal for the endowment to take on more risk to increase individual gifts.

Theorem 7. *Suppose felicity takes the CARA form and let $\tilde{x}$ follow a normal distribution with expectation* 0 *and variance σ^2. Then, $\lambda^* = 0$ and $g^*(\lambda^*) = \frac{1}{N+1}$.*

Of course, it is important to remember that while CARA felicity is popular for producing analytic solutions, it also implies an implausible attitude toward risk aversion.

For example, each coauthor of this paper is substantially less wealthy than billionaire Bill Gates. Yet, CARA discards the usual Inada condition (which predicts that marginal utility approaches infinity as wealth approaches zero). So, CARA predicts that each coauthors would hold the same *dollar* amount in risky assets as Bill Gates, by shorting the risk-free asset by, in our individual cases, quite sadly, a nearly similar amount!

6 Conclusions

This paper revisits the large literature on endowment investing but with a more complete micro-foundation that starts with a donor's objective function. Under a condition—which quickly converges to standard DARA preferences in the number of donors—we show that risk taking reduces donor free-riding, is Pareto improving, and is required by competition among endowments for donations, or, even more generally, by imperfect competition with complete information. Large endowments optimally take substantial risk even if donors are very averse to changes in endowment spending and expensive, risky investments don't outperform cash on average.

A strong cross-sectional "size effect" also emerges, where endowment risk taking increases in the size of the endowment. It is efficient for a large endowment to take on substantial risk even without the presence of fixed costs and even if the endowment does not have access to unique risk asset classes relative to its donors. In fact, risk taking is optimal even if the endowment's sponsors are very risk averse to changes in spending and use expensive investments that don't over-perform cash on average. The model, therefore, also shows that it is optimal for smaller endowments to take on less risk, even in the presence of more modern outsourced "endowment style" turnkey investment solutions.

Analogously to the Modigliani-Miller theorem, the focus of this paper is normative (deriving the optimal behavior of endowments facing many donors) rather than positive (describing how endowments actually behave). The normative focus lays a foundation for identification of agency issues. Indeed, our results challenge the conventional thinking about the relationship between endowment risk taking and agency. A *low* level of risk taking by a large endowment likely indicates a principal-agent conflict, where a non-diversified endowment investment manager, knowingly or unknowingly, exploits the incomplete information or bounded rationality of its donors.

The normative approach naturally lends itself to possible extensions, including understanding the potential value of organized capital campaigns that are often used in endowment fund raising. At first blush, capital campaigns would appear to give endowments an additional tool to reduce free-riding, even if a fairly weak one. At the same time, capital campaigns could mainly serve as economies of scale in messaging the charity's new set of common goals, which then becomes the main source of free-riding in giving. Measurement is potentially confounded by data limitations[34]as well as endogeneity problems, where charities facing greater free-riding are more likely to engage in multiple mechanisms to reduce it. Future work can explore this important issue in more detail.

References

ANDREONI, J. (1988): "Why free ride?: Strategies and learning in public goods experiments," *Journal of public Economics*, 37, 291–304.

——— (1998): "Toward a theory of charitable fund-raising," *Journal of Political Economy*, 106, 1186–1213.

ANG, A., A. AYALA, AND W. N. GOETZMANN (2018): "Investment beliefs of endowments," *European Financial Management*, 24, 3–33.

BAADE, R. A. AND J. O. SUNDBERG (1996): "What determines alumni generosity?" *Economics of Education Review*, 15, 75–81.

BECKER, G. S. (1976): "Altruism, egoism, and genetic fitness: Economics and sociobiology," *Journal of Economic Literature*, 14, 817–826.

BLACK, F. (1976): "The investment policy spectrum: individuals, endowment funds and pension funds," *Financial Analysts Journal*, 32, 23–31.

BROWN, J. R., S. G. DIMMOCK, J.-K. KANG, D. RICHARDSON, AND S. WEISBENNER (2011): "The governance of university endowments: Insights from a TIAA-CREF institute survey," *TIAA-CREF Institute Research Dialogue*, 102.

BROWN, J. R., S. G. DIMMOCK, J.-K. KANG, AND S. J. WEISBENNER (2014): "How university endowments respond to financial market shocks: Evidence and implications," *The American Economic Review*, 104, 931–62.

BROWN, J. R., S. G. DIMMOCK, AND S. WEISBENNER (2015): "The Supply of and Demand for Charitable Donations to Higher Education," in *How the Financial Crisis and Great*

[34]Major data trackers of university endowments—including Commonfund, NACUBO and VSE—do not appear to distinguish assets raised from capital campaigns from other assets. Indeed, most university assets are invested in so-called Long Term Pools.

Recession Affected Higher Education, ed. by J. R. Brown and C. M. Hoxby, University of Chicago Press, 151–174.

BROWN, J. R. AND D. W. WILCOX (2009): "Discounting state and local pension liabilities," *American Economic Review*, 99, 538–42.

BROWN, K. C. AND C. I. TIU (2015): "The Interaction of Spending Policies, Asset Allocation Strategies, and Investment Performance at University Endowment Funds," in *How the Financial Crisis and Great Recession Affected Higher Education*, ed. by J. R. Brown and C. M. Hoxby, University of Chicago Press, 43–98.

BUTCHER, K. F., C. KEARNS, AND P. J. MCEWAN (2013): "Giving till it helps? Alumnae giving and childrens college options," *Research in Higher Education*, 54, 499–513.

CEJNEK, G., R. FRANZ, O. RANDL, AND N. STOUGHTON (2014a): "A survey of university endowment management research," *Journal of Investment Management*, 12, 90–117.

CEJNEK, G., R. FRANZ, AND N. STOUGHTON (2014b): "An integrated model of university endowments," *WU Vienna University of Economics and Business Working Paper*.

CHAMBERS, D., E. DIMSON, AND J. FOO (2015): "Keynes, King's, and Endowment Asset Management," in *How the Financial Crisis and Great Recession Affected Higher Education*, ed. by J. R. Brown and C. M. Hoxby, University of Chicago Press, 127–150.

CLOTFELTER, C. T. (2003): "Alumni giving to elite private colleges and universities," *Economics of Education Review*, 22, 109–120.

CONSTANTINIDES, G. M. (1993): "Optimal Investment Strategies for University Endowment Funds: Comment," *Studies of Supply and Demand in Higher Education*, 236–242.

CONTI-BROWN, P. (2011): "Scarcity amidst wealth: The law, finance, and culture of elite university endowments in financial crisis," *Stanford Law Review*, 699–749.

CORE, J. E., W. R. GUAY, AND R. S. VERDI (2006): "Agency problems of excess endowment holdings in not-for-profit firms," *Journal of Accounting and Economics*, 41, 307–333.

DIMMOCK, S. G. (2012): "Background risk and university endowment funds," *Review of Economics and Statistics*, 94, 789–799.

DITTMANN, I., K.-C. YU, AND D. ZHANG (2017): "How important are risk-taking incentives in executive compensation?" *Review of Finance*, 21, 1805–1846.

DYBVIG, P. H. (1999): "Using asset allocation to protect spending," *Financial Analysts Journal*, 55, 49–62.

EECKHOUDT, L., C. GOLLIER, AND H. SCHLESINGER (2005): *Economic and Financial Decisions under Risk*, Princeton University Press.

EHRENBERG, RONALD G., J. L. C. AND J. EPIFANTSEVA (2001): "Paying our Presidents: What do Trustees Value?" *The Review of Higher Education*, 25, 15–37.

EHRENBERG, R. G. AND C. L. SMITH (2003): "The sources and uses of annual giving at selective private research universities and liberal arts colleges," *Economics of Education Review*, 22, 223–235.

FAMA, E. F. AND M. C. JENSEN (1985): "Organizational forms and investment decisions," *Journal of Financial Economics*, 14, 101–119.

FISMAN, R. AND R. G. HUBBARD (2005): "Precautionary savings and the governance of nonprofit organizations," *Journal of Public Economics*, 89, 2231–2243.

FLEISCHER, V. (2015): "Stop universities from hoarding money," *New York Times*.

GILBERT, T. AND C. HRDLICKA (2012): "Fairness and risk-sharing across generations: An application to university and nonprofit endowments," Tech. rep., University of Washington Working Paper.

——— (2013): "Why Do University Endowments Invest So Much In Risky Assets?" *Unpublished, University of Washington*.

GOETZMANN, W. N. AND S. OSTER (2015): "Competition among University Endowments," in *How the Financial Crisis and Great Recession Affected Higher Education*, ed. by J. R. Brown and C. M. Hoxby, University of Chicago Press, 99–126.

GOLLIER, C. (2004): *The Economics of Risk and Time*, MIT press.

HALL, B. J. AND K. J. MURPHY (2002): "Stock options for undiversified executives," *Journal of accounting and economics*, 33, 3–42.

HANSMANN, H. (1990): "Why do universities have endowments?" *The Journal of Legal Studies*, 19, 3–42.

HOXBY, C. M. (2015): "Endowment Management Based on a Positive Model of the University," in *How the Financial Crisis and Great Recession Affected Higher Education*, ed. by J. R. Brown and C. M. Hoxby, University of Chicago Press, 15–41.

JAMES, E. (1990): "Decision processes and priorities in higher education," *The Economics of American Universities*, 77–106.

KIMBALL, M. S. (1990): "Precautionary Saving in the Small and in the Large," *Econometrica*, 58, 53–73.

LEIBOWITZ, M. L., A. BOVA, AND P. B. HAMMOND (2010): *The Endowment Model of Investing: Return, Risk, and Diversification*, vol. 534, John Wiley & Sons.

LERNER, J., A. SCHOAR, AND J. WANG (2008): "Secrets of the academy: The drivers of university endowment success," *Journal of Economic Perspectives*, 22, 207–22.

LITVACK, J. M., B. G. MALKIEL, AND R. E. QUANDT (1974): "A plan for the definition of endowment income," *The American Economic Review*, 64, 433–437.

MEER, J. AND H. S. ROSEN (2009): "Altruism and the child cycle of alumni donations," *American Economic Journal: Economic Policy*, 1, 258–86.

MERTON, R. (1992): "Optimal Investment Strategies for University Endowment Funds." in *Continuous Time Finance–Revised Edition.*, Malden, MA: Blackwell Publishers.

MERTON, R. C. (1993): "Optimal investment strategies for university endowment funds," in *Studies of supply and demand in higher education*, ed. by C. T. Clotfelter and M. Rothschild, University of Chicago Press, 211–242.

MODIGLIANI, F. AND M. H. MILLER (1958): "The cost of capital, corporation finance and the theory of investment," *The American Economic Review*, 48, 261–297.

NCSE (2017): "NABUCO - Commonfund Study of Endowments," *Commonfund*.

NOVY-MARX, R. AND J. RAUH (2011): "Public pension promises: how big are they and what are they worth?" *The Journal of Finance*, 66, 1211–1249.

ROSE-ACKERMAN, S. (1996): "Altruism, nonprofits, and economic theory," *Journal of Economic Literature*, 34, 701–728.

ROSEN, H. S. AND A. J. SAPPINGTON (2016): "What do university endowment managers worry about? an analysis of alternative asset investments and background income," *Education Finance and Policy*, 11, 404–425.

SAMUELSON, P. A. (1954): "The pure theory of public expenditure," *The Review of Economics and Statistics*, 387–389.

STEINBERG, R. (1987): "Voluntary donations and public expenditures in a federalist system," *The American Economic Review*, 24–36.

SWENSEN, D. F. (2009): *Pioneering Portfolio Management: An Unconventional Approach to Institutional Investment, Fully Revised and Updated*, Simon and Schuster.

TOBIN, J. (1974): "What is permanent endowment income?" *The American Economic Review*, 64, 427–432.

Appendices

A Proofs

Theorem 1

Stage 2 For given donation vector $\vec{g}$ and investment level λ of the endowment fund, the optimal private investment decision vector $(\widehat{\alpha}_1^*, ..., \widehat{\alpha}_N^*)$ is the solution to the following maximization problem

$$\max_{\alpha_1,...,\alpha_N} \sum_{i=1}^{N} EU\left(\lambda, \vec{\alpha}, \vec{g}\right) = \sum_{i=1}^{N} E\left[u\left(1 + \alpha_i \tilde{x} - g_i\right)\right] + NE\left[v\left(\sum_{i=1}^{N} g_i + \lambda \tilde{x}\right)\right].$$

The FOCs are

$$\frac{\partial \sum_{i=1}^{N} EU\left(\lambda, \vec{\alpha}, \vec{g}\right)}{\partial \alpha_i} = E\left[\tilde{x} u'\left(1 + \alpha_i \tilde{x} - g_i\right)\right] = 0.$$

The SOCs are

$$\frac{\partial^2 \sum_{i=1}^{N} EU\left(\lambda, \vec{\alpha}, \vec{g}\right)}{\partial \alpha_i^2} = E\left[\tilde{x}^2 u''\left(1 + \alpha_i \tilde{x} - g_i\right)\right] < 0,$$

and thus satisfied. For $\alpha_i = 0$, we derive

$$\frac{\partial \sum_{i=1}^{N} EU\left(\lambda, \vec{\alpha}, \vec{g}\right)}{\partial \alpha_i} \mid_{\alpha_i = 0} = 0.$$

$\widehat{\alpha}_i^* = 0$ for all $i = 1, .., N$ is thus the unique global maximum.

For a given investment level λ of the endowment fund, the social planner picks the gift vector $(\widehat{g}_1^*, ..., \widehat{g}_N^*)$ to maximize

$$\sum_{i=1}^{N} EU_i\left(\lambda, \vec{\widehat{\alpha}}^* = \vec{0}, \vec{g}\right) = \sum_{i=1}^{N} u\left(1 - g_i\right) + NE\left[v\left(\sum_{i=1}^{N} g_i + \lambda \tilde{x}\right)\right].$$

The FOCs are

$$\frac{\partial \sum_{i=1}^{N} EU_i\left(\lambda, \vec{0}, \vec{g}\right)}{\partial g_i} = -u'\left(1 - g_i\right) + NE\left[v'\left(\sum_{i=1}^{N} g_i + \lambda \tilde{x}\right)\right] = 0.$$

The SOCs are satisfied since

$$\frac{\partial^2 \sum_{i=1}^N EU_i\left(\lambda,\vec{0},\vec{g}\right)}{\partial g_i^2} = u''\left(1-g_i\right) + NE\left[v''\left(\sum_{i=1}^N g_i + \lambda\tilde{x}\right)\right] < 0.$$

The unique optimal donation $\widehat{g}_i^*\left(\lambda\right)$ is thus the solution to the FOC

$$u'\left(1-\widehat{g}_i^*\left(\lambda\right)\right) = NE\left[v'\left(\sum_{i=1}^N \widehat{g}_i^*\left(\lambda\right) + \lambda\tilde{x}\right)\right].$$

This condition implies $\widehat{g}_1^*\left(\lambda\right) = ... = \widehat{g}_N^*\left(\lambda\right) = \widehat{g}^*\left(\lambda\right)$ and thus

$$u'\left(1-\widehat{g}^*\left(\lambda\right)\right) = NE\left[v'\left(N\widehat{g}^*\left(\lambda\right) + \lambda\tilde{x}\right)\right].$$

Stage 1 The optimal investment decision of the endowment fund $\widehat{\lambda}^*$ is the solution to the following maximization problem

$$\max_{\lambda} \Omega_{SO}\left(\lambda\right) = u\left(1-\widehat{g}^*\left(\lambda\right)\right) + E\left[v\left(N\widehat{g}^*\left(\lambda\right) + \lambda\tilde{x}\right)\right].$$

The FOC is

$$\Omega'_{SO}\left(\lambda\right) = -\widehat{g}^{*\prime}\left(\lambda\right)u'\left(1-\widehat{g}^*\left(\lambda\right)\right) + E\left[\left(N\widehat{g}^{*\prime}\left(\lambda\right) + \tilde{x}\right)v'\left(N\widehat{g}^*\left(\lambda\right) + \lambda\tilde{x}\right)\right] = 0.$$

Substitution of the FOC at Stage 2 yields

$$\Omega'_{SO}\left(\lambda\right) = E\left[\tilde{x}v'\left(N\widehat{g}^*\left(\lambda\right) + \lambda\tilde{x}\right)\right] = 0.$$

The concavity of $v\left(\cdot\right)$ implies

$$\begin{aligned} \Omega'_{SO}\left(\lambda\right) &> 0 \text{ for all } \lambda < 0, \\ \Omega'_{SO}\left(0\right) &= 0, \text{ and,} \\ \Omega'_{SO}\left(\lambda\right) &< 0 \text{ for all } \lambda > 0. \end{aligned}$$

Expected utility is thus globally concave in λ and $\widehat{\lambda}^* = 0$ is the unique global maximum. Last, the FOC for $\widehat{g}^*\left(\widehat{\lambda}^* = 0\right)$ then yields

$$u'\left(1-\widehat{g}^*\left(0\right)\right) = Nv'\left(N\widehat{g}^*\left(0\right)\right).$$

Note: If $u \equiv v$, then $\widehat{g}^*(0) > \frac{1}{N+1}$ as

$$u'\left(1 - \frac{1}{N+1}\right) < Nu'\left(N \cdot \frac{1}{N+1}\right)$$

and $u'' < 0$.

Theorem 2

Without loss of generality we consider investor 1. Given the donations of all other investors $\vec{g}_{-1} = (g_2, ..., g_N)$, their investment levels $\vec{\alpha}_{-1} = (\alpha_1, ..., \alpha_N)$, and the investment level λ of the endowment, investor 1's best response function $\alpha_1^*(\lambda, \vec{\alpha}_{-1}, \vec{g}_{-1})$ is given by the solution to the following maximization problem

$$\begin{aligned} \alpha_1^*(\lambda, \vec{\alpha}_{-1}, \vec{g}_{-1}) &\in \arg\max_{\alpha_1} EU_1(\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1}) \text{ with} \\ EU_1(\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1}) &= E\left[u\left(1 + \alpha_1 \tilde{x} - g_1\right)\right] + E\left[v\left(g_1 + \sum_{i=2}^{N} g_i + \lambda \tilde{x}\right)\right]. \end{aligned}$$

The FOC for the best response function is

$$\frac{\partial EU_1(\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1})}{\partial \alpha_1} = E\left[\tilde{x} u'\left(1 + \alpha_1 \tilde{x} - g_1\right)\right] = 0.$$

The SOC for the best response function holds as

$$\frac{\partial^2 EU_1(\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1})}{\partial \alpha_1^2} = E\left[\tilde{x}^2 u''\left(1 + \alpha_1 \tilde{x} - g_1\right)\right] < 0 \text{ for all } (\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1}).$$

Evaluating the FOC at $\alpha_1 = 0$ yields

$$\frac{\partial EU_1(\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1})}{\partial \alpha_1} \Big|_{\alpha_1 = 0} = 0 \text{ for all } (\lambda, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1}).$$

$\alpha_1^*(\lambda, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1}) = 0$ is thus the unique global maximum. Analogously, $\alpha_i^*(\lambda, \vec{\alpha}_{-i}, g_i, \vec{g}_{-i}) = 0$ for all i and it is thus the unique global maximum of the best response function of investor i. Thus, $\alpha_i^*(\lambda, g_i, \vec{g}_{-i}) = 0$ for all i is the unique Nash equilibrium for all $(\lambda, g_i, \vec{g}_{-i})$.

Given the donations $\vec{g}_{-1}$ of all other investors, investor 1's best response function

$g_1^* (\lambda, \vec{g}_{-1})$ is given by the solution to the following maximization problem

$$\begin{aligned} g_1^* (\lambda, \vec{g}_{-1}) &\in \arg\max_{g_1} EU_1 (\lambda, g_1, \vec{g}_{-1}) \text{ with} \\ EU_1 (\lambda, g_1, \vec{g}_{-1}) &= u(1 - g_1) + E\left[v\left(g_1 + \sum_{i=2}^{N} g_i + \lambda \tilde{x} \right) \right]. \end{aligned}$$

The FOC for the best response function is

$$\frac{\partial EU_1 (\lambda, g_1, \vec{g}_{-1})}{\partial g_1} = -u' (1 - g_1) + E\left[v'\left(g_1 + \sum_{i=2}^{N} g_i + \lambda \tilde{x} \right) \right] = 0.$$

The SOC for the best response function holds as

$$\frac{\partial^2 EU_1 (\lambda, g_1, \vec{g}_{-1})}{\partial g_1^2} = u'' (1 - g_1) + E\left[v''\left(g_1 + \sum_{i=2}^{N} g_i + \lambda \tilde{x} \right) \right] < 0.$$

Therefore, there exists a unique solution $g_1^* (\lambda, \vec{g}_{-1})$ to the above optimization problem which is determined by the FOC. We denote the Nash equilibrium by $(g_1^* (\lambda), ..., g_N^* (\lambda))$ which satisfies

$$\begin{aligned} u' (1 - g_1^* (\lambda)) &= u' (1 - g_i^* (\lambda)) \\ &= E\left[v'\left(\sum_{i=1}^{N} g_i^* (\lambda) + \lambda \tilde{x} \right) \right] \end{aligned}$$

for all i. Thus $g_1^* (\lambda) = ... = g_N^* (\lambda) = g^* (\lambda)$ which is given by the FOC

$$u' (1 - g^* (\lambda)) = E\left[v' (N g^* (\lambda) + \lambda \tilde{x}) \right].$$

Theorem 3

We derive the first- and second-order effects of changes in the investment policy λ on the Nash gift policy functions $g^* (\lambda)$. Implicitly differentiating the FOC for the Nash equilibrium with respect to λ yields

$$-g^{*\prime} (\lambda) u'' (1 - g^* (\lambda)) = E\left[\left(N g^{*\prime} (\lambda) + \tilde{x} \right) v'' (N g^* (\lambda) + \lambda \tilde{x}) \right],$$

i.e.,

$$g^{*\prime} (\lambda) = -\frac{E\left[\tilde{x} v'' (N g^* (\lambda) + \lambda \tilde{x}) \right]}{u'' (1 - g^* (\lambda)) + N E\left[v'' (N g^* (\lambda) + \lambda \tilde{x}) \right]}.$$

Evaluating this equation at $\lambda = 0$ yields $g^{*\prime}(0) = 0$. Taking the second deriative of the FOC of the Nash equilibrium with respect to λ yields

$$\begin{aligned} & -g^{*\prime\prime}(\lambda)\, u''(1 - g^*(\lambda)) + \left(g^{*\prime}(\lambda)\right)^2 u'''(1 - g^*(\lambda)) \\ = \;& Ng^{*\prime\prime}(\lambda)\, E\left[v''(Ng^*(\lambda) + \lambda\tilde{x})\right] + E\left[\left(Ng^{*\prime}(\lambda) + \tilde{x}\right)^2 v'''(Ng^*(\lambda) + \lambda\tilde{x})\right]. \end{aligned}$$

Evaluating this equation at $\lambda = 0$ yields

$$-g^{*\prime\prime}(0)\, u''(1 - g^*(0)) = Ng^{*\prime\prime}(0)\, v''(Ng^*(0)) + E\left[\tilde{x}^2\right] v'''(Ng^*(0)),$$

which implies

$$g^{*\prime\prime}(0) = -\frac{E\left[\tilde{x}^2\right] v'''(Ng^*(0))}{u''(1 - g^*(0)) + Nv''(Ng^*(0))}.$$

$\lambda = 0$ is a local minimum if and only if $g^{*\prime\prime}(0) > 0$. This holds if and only if $v'''(Ng^*(0)) > 0$ which is identical to the condition $P^v(Ng^*(0)) > 0$.

Now suppose $P^v(\cdot) > 0$. This implies

$$\begin{aligned} E\left[\tilde{x}v''(Ng^*(\lambda) + \lambda\tilde{x})\right] &< 0 \text{ for all } \lambda < 0, \text{ and,} \\ E\left[\tilde{x}v''(Ng^*(\lambda) + \lambda\tilde{x})\right] &> 0 \text{ for all } \lambda > 0, \end{aligned}$$

and thus

$$\begin{aligned} g^{*\prime}(\lambda) &< 0 \text{ for all } \lambda < 0, \\ g^{*\prime}(0) &= 0, \text{ and,} \\ g^{*\prime}(\lambda) &> 0 \text{ for all } \lambda > 0. \end{aligned}$$

$$\lambda = 0$$

is thus the global minimum of $g^*(\lambda)$.

Theorem 4

The endowment fund selects the optimal investment strategy λ^* by maximizing the expected utility of a single donor, $\Omega(\lambda)$. It is thus given by the solution to the following maximization problem

$$\lambda^* \in \arg\max_{\lambda} \Omega(\lambda) = u(1 - g^*(\lambda)) + E\left[v(Ng^*(\lambda) + \lambda\tilde{x})\right].$$

The first derivative is

$$\Omega'(\lambda) = -g^{*\prime}(\lambda)\,u'(1-g^*(\lambda)) + E\left[\left(Ng^{*\prime}(\lambda)+\tilde{x}\right)v'(Ng^*(\lambda)+\lambda\tilde{x})\right].$$

Substitution of the condition for the Nash equilibrium at Stage 2 yields

$$\Omega'(\lambda) = E\left[\left((N-1)\,g^{*\prime}(\lambda)+\tilde{x}\right)v'(Ng^*(\lambda)+\lambda\tilde{x})\right].$$

Evaluating this derivative at $\lambda = 0$ yields $\Omega'(0) = 0$.

The second derivative is given by

$$\begin{aligned}\Omega''(\lambda) &= (N-1)\,g^{*\prime\prime}(\lambda)\,E\left[v'(Ng^*(\lambda)+\lambda\tilde{x})\right]\\ &\quad +E\left[\left((N-1)\,g^{*\prime}(\lambda)+\tilde{x}\right)\left(Ng^{*\prime}(\lambda)+\tilde{x}\right)v''(Ng^*(\lambda)+\lambda\tilde{x})\right].\end{aligned}$$

Evaluating this second derivative at $\lambda = 0$ yields

$$\begin{aligned}\Omega''(0) &= (N-1)\,g^{*\prime\prime}(0)\,v'(Ng^*(0)) + E\left[\tilde{x}^2\right]v''(Ng^*(0))\\ &= -(N-1)\frac{E\left[\tilde{x}^2\right]v'''(Ng^*(0))}{u''(1-g^*(0))+Nv''(Ng^*(0))}v'(Ng^*(0)) + E\left[\tilde{x}^2\right]v''(Ng^*(0))\\ &= E\left[\tilde{x}^2\right]\left(-\frac{(N-1)\,v'''(Ng^*(0))\,v'(Ng^*(0))}{u''(1-g^*(0))+Nv''(Ng^*(0))}+v''(Ng^*(0))\right).\end{aligned}$$

$\lambda = 0$ is a local minimum if and only if $\Omega''(0) > 0$. With the FOC of the Nash equilibrium, $u'(1-g^*(0)) = v'(Ng^*(0))$, we derive that $\Omega''(0) > 0$ if and only if

$$(N-1)\,P^v(Ng^*(0)) > A^u(1-g^*(0)) + NA^v(Ng^*(0)).$$

Corollary 1. *If $u(\cdot) = v(\cdot)$, then $g^*(0) = \frac{1}{N+1}$ and $\lambda = 0$ is a local minimum if and only if*

$$P\left(\frac{N}{N+1}\right) > \frac{N+1}{N-1}\cdot A\left(\frac{N}{N+1}\right).$$

Theorem 5

The second inequality, $\Omega(\lambda^*) > \Omega(\lambda = 0)$ with $|\lambda^*| > 0$, is the result of Theorem 4. For the first inequality, note that the socially optimal $\widehat{\lambda}^* = 0$, while the Nash equilibrium implies $|\lambda^*| > 0$. (The value $\alpha_i = 0$ is optimal both socially and in the Nash equilibrium.)

We then have

$$\begin{aligned}\Omega_{SO}\left(\widehat{\lambda}^*=0\right) &= EU\left(\widehat{\lambda}^*=0,\widehat{g}^*\left(\widehat{\lambda}^*=0\right)\right)\\ &> EU\left(\left(\lambda^*,\widehat{g}^*\left(\lambda^*\right)\right)\right.\\ &> EU\left(\left(\lambda^*,g^*\left(\lambda^*\right)\right)\right.=\Omega\left(\lambda^*\right),\end{aligned}$$

where $|\lambda^*|>0$. The first inequality comes from knowing that the socially optimal value of $\widehat{\lambda}^*=0$ maximizes donor expected utility, equation (3). Hence, any other choice of $\lambda\neq 0$, including the value in the Nash equilibrium, λ^*, must produce a smaller expected utility, if inserted into the social optimal gift policy function. The second inequality follows from the fact that the social optimum problem maximizes donor expected utility, thereby producing a larger expected utility than in the Nash equilibrium, conditional on the same value of λ.

Theorem 6

Stage 2 The Nash equilibrium gifts $g^*(\lambda)$ are given by the condition

$$u'\left(1-g^*\left(\lambda\right)\right)=E\left[v'\left(Ng^*\left(\lambda\right)+\lambda\tilde{x}\right)\right].$$

Solving this condition for $u(w)=\ln(w)$ and $\tilde{x}$ following a two-point distribution that takes the values $+1$ and -1 with equal probability yields equation (10)

$$g^*\left(\lambda\right)=\frac{N+\sqrt{N^2+4N\left(N+1\right)\lambda^2}}{2N\left(N+1\right)}.$$

The first derivative of the gift policy function is

$$g^{*\prime}\left(\lambda\right)=\frac{2\lambda}{\sqrt{N^2+4N\left(N+1\right)\lambda^2}}.$$

Stage 1 The optimal investment strategy λ^* is given by the solution to the maximization problem

$$\lambda^*\in\arg\max_{\lambda}\Omega\left(\lambda\right)=u\left(1-g^*\left(\lambda\right)\right)+E\left[v\left(Ng^*\left(\lambda\right)+\lambda\tilde{x}\right)\right].$$

For $u(w)=\ln(w)$ and $\tilde{x}$ following a two-point distribution we derive

$$\Omega(\lambda) = \ln(1 - g^*(\lambda)) + \frac{1}{2}(\ln(Ng^*(\lambda) + \lambda) + (Ng^*(\lambda) - \lambda)).$$

Note that both $g^*(\lambda)$ and $\Omega(\lambda)$ are symmetric in λ. We thus focus on $\lambda \geq 0$. Furthermore, the domain restriction $\lambda < N$ ensures that $\Omega(\lambda)$ is well-defined, i.e. $1 - g^*(\lambda) > 0$ and $Ng^*(\lambda) - \lambda > 0$.

Substituting the condition for the Nash equilibrium gifts at Stage 2 into the first derivative yields

$$\begin{aligned}
\Omega'(\lambda) &= E\left[\left((N-1)g^{*\prime}(\lambda) + \tilde{x}\right)v'(Ng^*(\lambda) + \lambda\tilde{x})\right] \\
&= \frac{N(N-1)g^{*\prime}(\lambda)g^*(\lambda) - \lambda}{(Ng^*(\lambda) + \lambda)(Ng^*(\lambda) - \lambda)}.
\end{aligned}$$

The denominator is strictly positive. Solving the FOC $\Omega'(\lambda) = 0$ yields the solutions $\lambda = 0$ and $\lambda = \frac{\sqrt{N(N-3)}}{4}$.

Moreover, it can be shown that

$$\begin{aligned}
\Omega'(0) &= 0, \\
\Omega'(\lambda) &> 0 \text{ for all } 0 < \lambda < \frac{\sqrt{N(N-3)}}{4}, \\
\Omega'\left(\frac{\sqrt{N(N-3)}}{4}\right) &= 0, \text{ and,} \\
\Omega'(\lambda) &< 0 \text{ for all } \frac{\sqrt{N(N-3)}}{4} < \lambda < N.
\end{aligned}$$

Taking into account the symmetry of $\Omega(\lambda)$, we conclude that $\lambda = 0$ is a local minimum and the global maximum is attained at $|\lambda^*| = \frac{\sqrt{N(N-3)}}{4}$.

Evaluating the Nash equilibrium gifts at λ^* implies

$$g^*(\lambda^*) = \frac{1}{4}.$$

Theorem 7

Stage 2 The Nash equilibrium gifts $g^*(\lambda)$ are given by the condition

$$u'\left(1-g^*\left(\lambda\right)\right)=E\left[v'\left(Ng^*\left(\lambda\right)+\lambda\tilde{x}\right)\right].$$

Solving this condition for $u\left(w\right)=-e^{-\gamma w}$ and $\tilde{x}$ following a normal distribution with expectation 0 and variance σ^2 yields equation (11)

$$g^*\left(\lambda\right)=\frac{1}{N+1}\left(1+\frac{1}{2}\gamma\sigma^2\lambda^2\right).$$

Note that $E\left[e^{-\gamma\lambda\tilde{x}}\right]=e^{\frac{1}{2}\gamma^2\sigma^2\lambda^2}$. The first derivative of the gift policy function is

$$g^{*\prime}\left(\lambda\right)=\frac{\gamma\sigma^2\lambda}{N+1}.$$

Stage 1 The optimal investment strategy λ^* is given by the solution to the maximization problem

$$\lambda^*\in\arg\max_{\lambda}\Omega\left(\lambda\right)=u\left(1-g^*\left(\lambda\right)\right)+E\left[v\left(Ng^*\left(\lambda\right)+\lambda\tilde{x}\right)\right].$$

For $u\left(w\right)=-e^{-\gamma w}$ and $\tilde{x}$ following a normal distribution with expectation 0 and variance σ^2 we derive

$$\Omega\left(\lambda\right)=-2e^{-\frac{\gamma}{N+1}\left(N-\frac{1}{2}\gamma\sigma^2\lambda^2\right)}.$$

The first derivative yields

$$\Omega'\left(\lambda\right)=-\frac{2\gamma^2\sigma^2}{N+1}\lambda e^{-\frac{\gamma}{N+1}\left(N-\frac{1}{2}\gamma\sigma^2\lambda^2\right)}.$$

This implies

$$\begin{aligned}\Omega'\left(\lambda\right)&>0\text{ for all }\lambda<0,\\\Omega'\left(0\right)&=0\text{, and,}\\\Omega'\left(\lambda\right)&<0\text{ for all }\lambda>0.\end{aligned}$$

$\lambda^*=0$ is thus the unique maximum. Evaluating the Nash equilibrium gifts at $\lambda^*=0$ implies

$$g^*\left(\lambda^*\right)=\frac{1}{N+1}.$$